

UNITS IN INTEGRAL GROUP RINGS

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YUANLIN LI



Units in Integral Group Rings

by

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**Department of Mathematics and Statistics
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Abstract

In this thesis, we study several related problems concerning the unit group $\mathcal{U}(\mathbb{Z}G)$ of the integral group ring $\mathbb{Z}G$ of a periodic group G . Although Chapter 1 involves a lengthy calculation, the most important results appear in Chapter 2 through Chapter 5.

Chapter 1 describes constructively $\mathcal{U}(\mathbb{Z}(G \times C_2))$, where $\mathcal{U}(\mathbb{Z}G)$ has been described in some way. We are also interested in the following question: If G has a normal complement generated by bicyclic units, does $G \times C_2$ also have a normal complement generated by bicyclic units? We show that none of the normal complements of $D_8 \times C_2 \times C_2$ is generated by bicyclic units by explicitly constructing a set of generators for a normal complement of $D_8 \times C_2 \times C_2$, although a normal complement of $D_8 \times C_2$ is indeed generated by bicyclic units.

In chapter 2, we first study the subgroup of all unitary units $\mathcal{U}_f(\mathbb{Z}G)$. We prove that if G has a normal complement generated by unitary units, then it is also true for $G \times C_2$. Then we investigate generalized unitary units and prove that all of these units form a subgroup $\mathcal{U}_{g,f}(\mathbb{Z}G)$ of the unit group. Furthermore, we show that this subgroup is exactly the normalizer of the subgroup of unitary units. One of our main results is that the normalizer of $\mathcal{U}_{g,f}(\mathbb{Z}G)$ is equal to itself when G is a periodic group. We also obtain some other interesting results on $\mathcal{U}_{g,f}(\mathbb{Z}G)$.

Chapter 3 investigates central units of $\mathbb{Z}A_5$. We show that the centre $\mathcal{C}(\mathcal{U}(\mathbb{Z}A_5)) = \pm \langle u \rangle$, where $\langle u \rangle$ is an infinite cyclic group and we explicitly find the generator u .

In chapter 4, we study the hypercentral units in the integral group ring of a

periodic group G . We prove that the central height is at most 2. We also discuss the relationship between hypercentral units and generalized unitary units.

Chapter 5 characterizes the n -centre of the unit group of the integral group ring of a periodic group. It is proved that the n -centre is either the centre or the second centre of the unit group for all $n \geq 2$.

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Contents

Abstract	ii
Acknowledgement	iv
Introduction	1
1 Units of $\mathbb{Z}(D_8 \times C_2)$ and $\mathbb{Z}(D_8 \times C_2 \times C_2)$	5
1.1 Preliminaries and Notations	6
1.2 Units of $\mathbb{Z}(G \times C_2)$	10
1.3 Description of $\mathcal{U}(\mathbb{Z}(D_8 \times C_2))$	11
1.3.1 Machinery	13
1.3.2 Generators of the Normal Complement W	19
1.3.3 Bicyclic Units in $\mathcal{U}(\mathbb{Z}(D_8 \times C_2))$	22
1.4 Description of $\mathcal{U}(\mathbb{Z}(D_8 \times C_2 \times C_2))$	24
1.4.1 A Matrix Presentation of the Normal Complement	25
1.4.2 Generators for the Normal Complement	34
1.4.3 Bicyclic Units in $\mathcal{U}(\mathbb{Z}(D_8 \times C_2 \times C_2))$	41
1.5 Units in Other Integral Group Rings	45

2	Unitary and Generalized Unitary Units in Integral Group Rings	46
2.1	Unitary Units in Integral Group Rings	47
2.2	Generalized Unitary Units	54
2.2.1	The Normalizers of \mathcal{U}_f	54
2.2.2	Conditions for $\mathcal{U} = \mathcal{U}_{g,f}$	59
2.2.3	Conditions for $\mathcal{U}_{g,f} = \mathcal{U}_f$	64
2.2.4	Other Results	67
3	Central Units	72
3.1	Introduction	72
3.2	Main Results	76
4	Hypercentral Units in the Integral Group Ring of a Periodic Group	83
4.1	Introduction	83
4.2	Main Results	84
4.3	The Relationship Between Hypercentral Units and Generalized Unitary Units	92
5	The N-Centre of the Unit Group of an Integral Group Ring	95
5.1	Introduction	95
5.2	Basic Results and Notations	96
5.3	The Characterization of the N-Centre of the Unit Group of an Integral Group Ring	98
	Bibliography	107

Introduction

The group ring RG of the group G over a commutative unital ring R , a ring with the elements of G as a basis and with multiplication defined distributively using the group multiplication in G , reflects properties of the group G and the ring of coefficients R . Since this fascinating object is at the cross roads of several mathematical topics such as group theory, representation theory, number theory, and ring theory, it can be studied not only for its own sake but also as a tool for tackling other mathematical problems. The theory of group rings as an independent area of study has developed only in relatively recent times, following the fundamental work of Higman [21, 22], and it gained great impetus after the inclusion of questions on group rings in Kaplansky's famous lists of problems [31, 32]. The area was further stimulated by the inclusion of sections on group rings in the books on ring theory by Lambek [38] and Ribenboim [51]. Since then several books devoted entirely to the subject have appeared (e.g. Passi [47], Passman [48, 49] and Sehgal [57, 58]).

The unit group $\mathcal{U}(RG)$, consisting of all invertible elements in RG , plays a very important role in studying the relation between the group-theoretic structure of G and its group ring RG . Early significant results about the unit group were proved by Higman [21, 22], and since then considerable work has been done on this subject. In the important case where $R = \mathbb{Z}$, the ring of rational integers, the most important reference is the comprehensive book by Sehgal [58].

In the study of group rings, one very fundamental problem is describing $\mathcal{U}(\mathbb{Z}G)$ in some concrete way. A complete description (including generators) has been carried out for only a relatively small number of groups. The unit group $\mathcal{U}(\mathbb{Z}S_3)$ was

first described by Hughes and Pearson [24] and a different description was given by Allen and Hobby [2]. Most recently, Jespers and Parmenter [28] gave another characterization of $\mathcal{U}(\mathbb{Z}S_3)$, which allows us to obtain additional information about the structure of this unit group. The structure of $\mathcal{U}(\mathbb{Z}D_8)$ was first obtained by Polcino Milies [40]. A more recent result of Jespers and Leal [27] (see also Parmenter[44]) shows the freeness of a normal complement and also the important role played by the bicyclic units. Most recently Jespers and Parmenter [29] extended the description of $\mathcal{U}(\mathbb{Z}G)$ for groups of order 16 and highlighted the important role played by the bicyclic units and Bass cyclic units. Some other examples of concrete descriptions of $\mathcal{U}(\mathbb{Z}G)$ include Allen-Hobby [1] for S_4 and A_4 , Passman-Smith [50] for D_{2p} , Galovitch-Reiner-Ullom [56] for $G = C_p \rtimes C_q$ where q is a prime dividing $p - 1$, Ritter-Sehgal [52] for $|G| = p^3$, Kleinert [35, 36] for $G = D_{2n}$ where n is an odd number which is a product of distinct primes, and for Q_p , the generalized quaternion group.

We will shed further light on this subject in Chapter 1. We give matrix presentations of a normal complement for the following groups: $D_8 \times C_2$, $D_8 \times C_2 \times C_2$, $D_6 \times C_2$, D_{10} and D_{14} . We are mainly interested in describing constructively $\mathcal{U}(Z(G \times C_2))$, where $\mathcal{U}(\mathbb{Z}G)$ has been described in some way. For example, we consider the following question: If G has a normal complement generated by bicyclic units, does $G \times C_2$ also have a normal complement generated by bicyclic units? We give a negative answer to this question, showing that none of the normal complements of $D_8 \times C_2 \times C_2$ is generated by bicyclic units, although a normal complement of $D_8 \times C_2$ is indeed generated by bicyclic units.

In Chapter 2, we first study the subgroup of all unitary units $\mathcal{U}_f(\mathbb{Z}G)$ (section

2.1). The study of unitary units was proposed by Novikov [43] and this subgroup was first described by Bovdi [10]. Since then a number of interesting results on this subject have appeared [11, 12, 13, 14, 15, 16, 23, 45]. Continuing on with the investigation initiated in Chapter 1, we establish a relationship between unitary units in $\mathbb{Z}G$ and $\mathbb{Z}(G \times C_2)$ and we also characterize when $\mathcal{U}_f(\mathbb{Z}G)$ is a subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$. Then we introduce and investigate generalized unitary units in the rest of the chapter. We show that all of these units form a subgroup $\mathcal{U}_{g,f}(\mathbb{Z}G)$ of the unit group which is exactly the normalizer of the subgroup of unitary units. One of our main results is that the normalizer of $\mathcal{U}_{g,f}(\mathbb{Z}G)$ is equal to itself when G is a periodic group. Among other results, we give necessary and sufficient conditions for the unit group to be generalized unitary when G is periodic and also characterize when all bicyclic units are nontrivial and generalized unitary.

Central units of integral group rings play a very important role in the study of generalized unitary units in Chapter 2. However, there are very few cases known of nonabelian groups G where the group of central units of $\mathbb{Z}G$, denoted $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$, is nontrivial and where the structure of $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$, including a complete set of generators, has been determined. In Chapter 3, we show that the central units of augmentation 1 in the integral group ring $\mathbb{Z}A_5$ form an infinite cyclic group $\langle u \rangle$, and we explicitly find the generator u .

After studying central units, it is natural to consider the hypercentral units. In Chapter 4, we study the hypercentral units in the integral group ring of a periodic group G . We prove that the central height is at most 2. This extends work of Arora, Hales and Passi [3], who proved the same result for finite groups. We also discuss the relationship between hypercentral units and generalized unitary units.

Another extension of the centre $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ is the n -centre, introduced by Baer [5]. It shares many properties with the centre, for example it follows from Corollary 1 in Baer [6] that a group is n -abelian if the quotient modulo its n -centre is (locally) cyclic. In [33], Kappe and Newell shed further light on these similarities by investigating various characterizations and embedding properties of the n -centre. Our main result in Chapter 5 is a complete characterization of the n -centre of the unit group of the integral group ring of a periodic group. To be specific, we prove that the n -centre is either the centre or the second centre of the unit group for all $n \geq 2$.

Chapter 1

Units of

$\mathbb{Z}(D_8 \times C_2)$ and $\mathbb{Z}(D_8 \times C_2 \times C_2)$

Let G be a finite group, $\mathcal{U}(\mathbb{Z}G)$ the group of units of the integral group ring $\mathbb{Z}G$ and $\mathcal{U}_1(\mathbb{Z}G)$ the subgroup of units of augmentation 1. Higman has a very famous theorem [22]:

Theorem 1.0.1. *For a finite group G , $\mathcal{U}(\mathbb{Z}G) = \pm G$ if and only if G is abelian of exponent 1, 2, 3, 4, 6 or $G = E \times Q_8$ where Q_8 is the quaternion group of order 8 and E is an elementary abelian 2-group.*

One way of proving this is to first show that $\mathcal{U}(\mathbb{Z}Q_8)$ is trivial - i.e. $\mathcal{U}(\mathbb{Z}Q_8) = \pm Q_8$, and to next prove that if $\mathcal{U}(\mathbb{Z}G)$ is trivial, then $\mathcal{U}(\mathbb{Z}(G \times C_2))$ is also trivial. Motivated by the latter result, in this chapter we are primarily concerned with the problem of describing constructively $\mathcal{U}(\mathbb{Z}(G \times C_2))$ for particular groups G , where $\mathcal{U}(\mathbb{Z}G)$ has been described in some way. We are also interested in the following question: If G has a normal complement generated by bicyclic units, does $G \times C_2$ also have a normal complement generated by bicyclic units?

Section 1.1 introduces preliminaries and notations, while section 1.2 theoretically

gives a general method of going from generators of $\mathcal{U}(\mathbb{Z}G)$ to generators of $\mathcal{U}(\mathbb{Z}(G \times C_2))$. Sections 1.3 and 1.4 describe all units of the integral group rings of groups $D_8 \times C_2$ and $D_8 \times C_2 \times C_2$ respectively. Section 1.4 also gives a negative answer to the question proposed earlier, and section 1.5 deals with units of other integral group rings.

1.1 Preliminaries and Notations

Describing the unit group $\mathcal{U}(\mathbb{Z}G)$ of the integral group ring $\mathbb{Z}G$ of a finite group is a fascinating and fundamental part of the study of group rings. A complete description (including generators) has been carried out for only a relatively small number of groups (see Sehgal [57, 58] for an excellent survey, as well as the introduction).

In [26], Jespers studied $\mathcal{U}(\mathbb{Z}(D_8 \times C_2))$ and $\mathcal{U}(\mathbb{Z}(S_3 \times C_2))$ and proved that $D_8 \times C_2$ as well as $S_3 \times C_2$ has a torsion-free normal complement which is generated by bicyclic units. This torsion-free normal complement of $D_8 \times C_2$ was described as a semi-direct product of free groups. We will give a matrix presentation of such a normal complement and construct fewer bicyclic generators for it (in section 1.3). Our description of $\mathcal{U}(\mathbb{Z}(D_8 \times C_2))$ is required in studying the group $\mathcal{U}(\mathbb{Z}(D_8 \times C_2 \times C_2))$ whose generators have not been constructed before (in section 1.4).

Let us recall some basic definitions and fundamental results which will be needed later in Chapter 1.

(i) Bicyclic Units;

Let $a, b \in G$, where $o(b)$ is finite. We write \hat{b} for the sum of all powers of b :

$$\hat{b} = \sum_1^{o(b)} b^i$$

Then $(1-b)\hat{b} = 0$ and for any $a \in G$, $((1-b)a\hat{b})^2 = 0$. Hence $u_{b,a} = 1 + (1-b)a\hat{b}$ has inverse $1 - (1-b)a\hat{b}$. The units $u_{b,a}$, $a, b \in G$ are called bicyclic units of $\mathbb{Z}G$.

(ii) Normal Complement;

If H is a subgroup of a group G , then H has a normal complement N if $N \triangleleft G$, $HN = G$ and $H \cap N = 1$. In subsequent sections, we will be primarily interested in a particular torsion-free normal complement of G in $\mathcal{U}_1(\mathbb{Z}G)$. We will need the following theorem about normal complements:

Theorem 1.1.1. *(See Sehgal [58], p.160, Theorem(31.1)) Let G be a finite group having an abelian normal subgroup A , such that either*

(a) G/A is abelian of exponent dividing 4 or 6 or

(b) G/A is abelian of odd order.

Then G has a normal torsion-free complement in $\mathcal{U}_1(\mathbb{Z}G)$.

In case (a), a normal complement to G in $\mathcal{U}_1(\mathbb{Z}G)$ is always given by $\mathcal{U}(1 + \Delta(G)\Delta(A))$. Case (b) is more difficult (recall that if H is a group, $\Delta(H)$ is the augmentation ideal of $\mathbb{Z}H$, namely $\{\sum \alpha_h h \mid \sum \alpha_h = 0\}$).

(iii) Schreier Method;

Theorem 1.1.2. (See Passman [49], p.117, Lemma 1.7) *Let G be a finitely generated group, and let H be a subgroup of finite index. Then H is finitely generated.*

The proof of this theorem provides a method, called the Schreier method, of finding generators of the subgroup from those of the group. We describe it briefly: let $G = \langle x_1, x_2, \dots, x_t \rangle$, and let $\{y_1, y_2, \dots, y_n\}$ be a complete set of right coset representatives for H in G . Let $h_{ij} = y_i x_j y_i^{-1}$, where y_i^{-1} is chosen such that $h_{ij} \in H$. Then H is generated by the finitely many elements h_{ij} , where $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, t$.

(iv) *PSL*

- Definition of $GL(n, \mathbb{Z}), SL(n, \mathbb{Z})$

The group of all invertible matrices $M_{n \times n}(\mathbb{Z})$ is called “the general linear group” and denoted by $GL(n, \mathbb{Z})$. The subgroup of $GL(n, \mathbb{Z})$ consisting of all matrices of determinant 1 is called “the special linear group” and denoted by $SL(n, \mathbb{Z})$.

- Definition of $PGL(n, \mathbb{Z}), PSL(n, \mathbb{Z})$

$$PGL(n, \mathbb{Z}) = GL(n, \mathbb{Z}) / \{\pm I_n\},$$

$$PSL(n, \mathbb{Z}) = SL(n, \mathbb{Z}) / \{\pm I_n\}.$$

where I_n is the $n \times n$ identity matrix.

- Definition of $\Gamma(n)$

We write Γ for $PSL(2, \mathbb{Z})$. Letting $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a typical element of Γ , we define the principal congruence subgroup of Γ of level n , denoted by $\Gamma(n)$, as the subset of Γ such that

$$a \equiv d \equiv 1 \pmod{n}, \quad b \equiv c \equiv 0 \pmod{n}.$$

We have the following important results:

Lemma 1.1.3. (See Newman [41], p.146-147) $\Gamma(n)$ is a subgroup of finite index of Γ for all n and the index is shown as follows:

$$\mu(n) = (\Gamma : \Gamma(n)) = \begin{cases} \frac{1}{2}n^3 \prod_{p|n} (1 - \frac{1}{p^2}) & n > 2 \\ 6 & n = 2 \end{cases}$$

The first few values of $\mu(n)$ are as follows:

n	1	2	3	4	5
μ	1	6	12	24	60

- Generators for $\Gamma(2)$

$\Gamma(2)$ is a free group of rank 2 and has $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ as free generators (Newman [41], p.144, Theorem VIII.7, and p.149).

(v) Dihedral Group D_{2n}

The finite group $D_{2n} = \langle a, b \mid a^n = b^2 = 1, ba = a^{n-1}b \rangle$ is called the dihedral group of order $2n$. Recall that D_8 was mentioned earlier and that D_6 is isomorphic to S_3 .

1.2 Units of $\mathbb{Z}(G \times C_2)$

In this section, we describe $\mathcal{U}(\mathbb{Z}(G \times C_2))$ in terms of $\mathcal{U}(\mathbb{Z}G)$. Theoretically, this section gives a general method of finding generators of $\mathcal{U}(\mathbb{Z}(G \times C_2))$ from those of $\mathcal{U}(\mathbb{Z}G)$.

Let G be a group and $C_2 = \langle c \rangle$ be a cyclic group of order 2.

Define $f_1 : \mathbb{Z}(G \times C_2) \rightarrow \mathbb{Z}G$ by $f_1(\sum \alpha_i g_i + \sum \beta_i g_i c) = \sum (\alpha_i + \beta_i) g_i$, where $\alpha_i, \beta_i \in \mathbb{Z}, g_i \in G$ for all i . Then f_1 is a homomorphism.

Define $f_2 : \mathbb{Z}(G \times C_2) \rightarrow \mathbb{Z}G$ by $f_2(\sum \alpha_i g_i + \sum \beta_i g_i c) = \sum (\alpha_i - \beta_i) g_i$. Then f_2 is a homomorphism.

Define $f : \mathbb{Z}(G \times C_2) \rightarrow \mathbb{Z}G \times \mathbb{Z}G$ by $f = (f_1, f_2)$. Then $\text{Ker}(f) = 0$, so that f is a monomorphism. It is easy to see

$$\text{Im}(f) = \{(\sum \gamma_i g_i, \sum \varepsilon_i g_i) \mid \gamma_i \equiv \varepsilon_i \pmod{2} \text{ for all } i\}.$$

Certainly, if $u \in \mathcal{U}(\mathbb{Z}(G \times C_2))$, then $f(u) \in V = \text{Im}(f) \cap (\mathcal{U}(\mathbb{Z}G) \times \mathcal{U}(\mathbb{Z}G))$. But conversely, assume $v = (\sum \gamma_i g_i, \sum \varepsilon_i g_i) \in V$, i.e. $\sum \gamma_i g_i \in \mathcal{U}(\mathbb{Z}G)$, $\sum \varepsilon_i g_i \in \mathcal{U}(\mathbb{Z}G)$ and $\gamma_i \equiv \varepsilon_i \pmod{2}$ for all i . Mapping into $\mathbb{Z}_2 G$, we obtain $\sum \bar{\gamma}_i g_i = \sum \bar{\varepsilon}_i g_i \in \mathcal{U}(\mathbb{Z}_2 G)$, thus $(\sum \bar{\gamma}_i g_i)^{-1} = (\sum \bar{\varepsilon}_i g_i)^{-1} \in \mathcal{U}(\mathbb{Z}_2 G)$.

If $(\sum \gamma_i g_i)^{-1} = \sum \theta_i g_i$ and $(\sum \varepsilon_i g_i)^{-1} = \sum y_i g_i$, then $\theta_i \equiv y_i \pmod{2}$ for all i . Therefore, $v^{-1} = (\sum \theta_i g_i, \sum y_i g_i) \in \text{Im}(f)$ and the preimage of it is exactly the inverse of the preimage of v . This shows that $f(\mathcal{U}(\mathbb{Z}(G \times C_2))) = \text{Im}(f) \cap (\mathcal{U}(\mathbb{Z}G) \times \mathcal{U}(\mathbb{Z}G))$.

We have proved that

Theorem 1.2.1. $\mathcal{U}(\mathbb{Z}(G \times C_2)) \cong \{v = (\sum \gamma_i g_i, \sum \varepsilon_i g_i) \in \mathcal{U}(\mathbb{Z}G) \times \mathcal{U}(\mathbb{Z}G) \mid \gamma_i \equiv \varepsilon_i \pmod{2} \text{ for all } i\} \subset \mathcal{U}(\mathbb{Z}G) \times \mathcal{U}(\mathbb{Z}G)$.

Corollary 1.2.2. *If $\mathcal{U}(\mathbb{Z}G)$ is trivial, then $\mathcal{U}(\mathbb{Z}(G \times C_2))$ is trivial.*

Remark 1.2.3. *V is a subgroup of finite index of $\mathcal{U}(\mathbb{Z}G) \times \mathcal{U}(\mathbb{Z}G)$ when G is finite.*

Proof. Let $A, B \in \mathcal{U}(\mathbb{Z}G) \times \mathcal{U}(\mathbb{Z}G)$, where $A = (\sum \alpha_i g_i, \sum \beta_i g_i)$, $B = (\sum w_i g_i, \sum y_i g_i)$ and $\alpha_i \equiv w_i \pmod{2}, \beta_i \equiv y_i \pmod{2}$. Mapping A, B into $\mathbb{Z}_2 G \times \mathbb{Z}_2 G$, we obtain $\bar{A} = \bar{B}$, thus $\bar{A}\bar{B}^{-1} = 1$. Therefore, $AB^{-1} = 1 + 2(\sum \alpha'_i g_i, \sum \beta'_i g_i) \in V$ and A, B are in the same coset of V . We point out that an upper bound of the index is $2^{2|G|}$. \square

If a complete set of coset representatives could be found, the Schreier method would now give us a way of going from generators of $\mathcal{U}(\mathbb{Z}G)$ to generators of $\mathcal{U}(\mathbb{Z}(G \times C_2))$. In practice, however, it seems very difficult even to decide the index, which may be very large, and still more difficult to find a complete set of coset representatives for the subgroup.

Instead we will use an alternative method to describe unit groups for the following examples. Our main goal is to compute generators for $\mathcal{U}(\mathbb{Z}(D_8 \times C_2 \times C_2))$. For this purpose, we need first deal with $\mathcal{U}(\mathbb{Z}(D_8 \times C_2))$.

1.3 Description of $\mathcal{U}(\mathbb{Z}(D_8 \times C_2))$

In this section, we give a new description of $\mathcal{U}(\mathbb{Z}(D_8 \times C_2))$, which allows us to obtain additional information about the structure of this unit group. Furthermore, we construct a set of bicyclic generators for a torsion-free normal complement of $D_8 \times C_2$ in $\mathcal{U}_1(\mathbb{Z}(D_8 \times C_2))$. The techniques and results developed here will be used in subsequent sections.

Our main result is as follows:

Theorem 1.3.1. *In $\mathcal{U}_1(\mathbb{Z}(D_8 \times C_2))$, $D_8 \times C_2$ has a torsion-free normal complement $W = \{u = 1 + \alpha(1 - a^2) | \alpha \in \Delta(D_8 \times C_2), u \text{ a unit}\}$, which is generated by bicyclic units. More explicitly,*

$$W \cong \left\{ \left(\begin{bmatrix} 1 + 4w_{11} & 4w_{12} \\ 2w_{21} & 1 + 4w_{22} \end{bmatrix}, \begin{bmatrix} 1 + 4z_{11} & 4z_{12} \\ 2z_{21} & 1 + 4z_{22} \end{bmatrix} \right) \mid \begin{matrix} 2|w_{12} + z_{12} & 2|w_{21} + z_{21} \\ w_{ij}, z_{ij} \in \mathbb{Z} & \det = 1 \end{matrix} \right\}$$

where “ $\det = 1$ ” means that the matrix in each component has determinant 1 and this will be used throughout Chapter 1.

Let $D_8 \times C_2 = \langle a, b, c | a^4 = b^2 = c^2 = 1, ba = a^3b, ac = ca, \text{ and } bc = cb \rangle$.

Note that if we let $A = \langle 1, a^2 \rangle$, then $(D_8 \times C_2)/A \cong C_2 \times C_2 \times C_2$; therefore, by Theorem 1.1.1, $D_8 \times C_2$ has a torsion-free normal complement $W = \mathcal{U}(1 + \Delta(D_8 \times C_2) \triangle (A)) = \{u = 1 + \alpha(1 - a^2) | \alpha \in \Delta(D_8 \times C_2) \text{ and } u \text{ a unit}\}$, so that the first statement holds. Next notice that a typical element in $\mathbb{Z}(D_8 \times C_2)(1 - a^2)$ can be written as

$$(\beta + \gamma c)(1 - a^2) = (\beta - \gamma)(1 - a^2)\left(\frac{1 - c}{2}\right) + (\beta + \gamma)(1 - a^2)\left(\frac{1 + c}{2}\right)$$

where $\beta, \gamma \in \mathbb{Z}D_8$.

As a consequence,

$$\begin{aligned} \alpha(1 - a^2) \in \mathbb{Z}(D_8 \times C_2)(1 - a^2) &\subseteq \mathbb{Z}D_8(1 - a^2)\left(\frac{1 - c}{2}\right) \oplus \mathbb{Z}D_8(1 - a^2)\left(\frac{1 + c}{2}\right) \\ &\subseteq \mathbb{Q}D_8(1 - a^2)\left(\frac{1 - c}{2}\right) \oplus \mathbb{Q}D_8(1 - a^2)\left(\frac{1 + c}{2}\right). \end{aligned}$$

Therefore, we only need to deal with $\mathbb{Q}D_8(1 - a^2)\left(\frac{1 - c}{2}\right)$ and $\mathbb{Q}D_8(1 - a^2)\left(\frac{1 + c}{2}\right)$.

1.3.1 Machinery

We first examine carefully the Wedderburn decomposition of the rational group algebra $Q(D_8 \times C_2)$. Consider the idempotents $f_1 = \frac{1-a^2}{2} \frac{1-c}{2}$ and $f_2 = \frac{1-a^2}{2} \frac{1+c}{2}$. Then

$$Q(D_8 \times C_2)f_1 = \mathbb{Q}D_8f_1 \cong M_{2 \times 2}(Q)$$

and

$$Q(D_8 \times C_2)f_2 = \mathbb{Q}D_8f_2 \cong M_{2 \times 2}(Q)$$

and elementary matrix bases for $Q(D_8 \times C_2)f_1$ and $Q(D_8 \times C_2)f_2$ over Q are given as follows:

$$\begin{aligned} e_{11} &= \frac{1+b}{2} f_1 \\ e_{12} &= \frac{ab-a}{2} f_1 \\ e_{21} &= \frac{ab+a}{2} f_1 \\ e_{22} &= \frac{1-b}{2} f_1 \end{aligned}$$

and

$$\begin{aligned} s_{11} &= \frac{1+b}{2} f_2 \\ s_{12} &= \frac{ab-a}{2} f_2 \\ s_{21} &= \frac{ab+a}{2} f_2 \\ s_{22} &= \frac{1-b}{2} f_2 \end{aligned}$$

As we pointed out before, $\alpha(1-a^2) \in \mathbb{Z}(D_8 \times C_2) \subseteq \mathbb{Q}D_8(1-a^2) \frac{1-c}{2} \oplus \mathbb{Q}D_8(1-a^2) \frac{1+c}{2} \cong M_{2 \times 2}(Q) \oplus M_{2 \times 2}(Q)$. We wish to see exactly what this embedding looks like with regard to the given elementary matrix bases.

Let us consider a typical element of $\mathbb{Z}(D_8 \times C_2)(1 - a^2)$, say, $v = (\alpha_0 + \alpha_1 a + \alpha_2 b + \alpha_3 ab + \beta_0 c + \beta_1 ac + \beta_2 bc + \beta_3 abc)(1 - a^2)$, where all α 's and β 's are in \mathbb{Z} . It can be rewritten in the form

$$\begin{aligned} v &= (\alpha_0 + \alpha_1 a + \alpha_2 b + \alpha_3 ab + \beta_0 c + \beta_1 ac + \beta_2 bc + \beta_3 abc)(1 - a^2) \frac{1 - c}{2} \\ &\quad + (\alpha_0 + \alpha_1 a + \alpha_2 b + \alpha_3 ab + \beta_0 c + \beta_1 ac + \beta_2 bc + \beta_3 abc)(1 - a^2) \frac{1 + c}{2} \\ &= 2((\alpha_0 - \beta_0) + (\alpha_1 - \beta_1)a + (\alpha_2 - \beta_2)b + (\alpha_3 - \beta_3)ab) \frac{1 - a^2}{2} \frac{1 - c}{2} \\ &\quad + 2((\alpha_0 + \beta_0) + (\alpha_1 + \beta_1)a + (\alpha_2 + \beta_2)b + (\alpha_3 + \beta_3)ab) \frac{1 - a^2}{2} \frac{1 + c}{2}. \end{aligned}$$

We have

$$\begin{aligned} f_1 &= e_{11} + e_{22} \\ af_1 &= e_{21} - e_{12} \\ bf_1 &= e_{11} - e_{22} \\ abf_1 &= e_{12} + e_{21} \end{aligned}$$

Similarly,

$$\begin{aligned} f_2 &= s_{11} + s_{22} \\ af_2 &= s_{21} - s_{12} \\ bf_2 &= s_{11} - s_{22} \\ abf_2 &= s_{12} + s_{21} \end{aligned}$$

Hence the corresponding pair of matrices is

$$\left(\begin{bmatrix} 2(\alpha_0 + \alpha_2 - \beta_0 - \beta_2) & 2(-\alpha_1 + \alpha_3 + \beta_1 - \beta_3) \\ 2(\alpha_1 + \alpha_3 - \beta_1 - \beta_3) & 2(\alpha_0 - \alpha_2 - \beta_0 + \beta_2) \end{bmatrix}, \right. \\ \left. \begin{bmatrix} 2(\alpha_0 + \alpha_2 + \beta_0 + \beta_2) & 2(-\alpha_1 + \alpha_3 - \beta_1 + \beta_3) \\ 2(\alpha_1 + \alpha_3 + \beta_1 + \beta_3) & 2(\alpha_0 - \alpha_2 + \beta_0 - \beta_2) \end{bmatrix} \right)$$

Conjugating both by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, we obtain

$$\left(\begin{bmatrix} 2[(\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3) - (\beta_0 - \beta_1 + \beta_2 - \beta_3)] & 4[(-\alpha_1 + \alpha_2) - (-\beta_1 + \beta_2)] \\ 2[(\alpha_1 + \alpha_3) - (\beta_1 + \beta_3)] & 2[(\alpha_0 + \alpha_1 - \alpha_2 + \alpha_3) - (\beta_0 + \beta_1 - \beta_2 + \beta_3)] \end{bmatrix}, \right. \\ \left. \begin{bmatrix} 2[(\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3) + (\beta_0 - \beta_1 + \beta_2 - \beta_3)] & 4[(-\alpha_1 + \alpha_2) + (-\beta_1 + \beta_2)] \\ 2[(\alpha_1 + \alpha_3) + (\beta_1 + \beta_3)] & 2[(\alpha_0 + \alpha_1 - \alpha_2 + \alpha_3) + (\beta_0 + \beta_1 - \beta_2 + \beta_3)] \end{bmatrix} \right)$$

This is a pair of matrices of the form

$$\left(\begin{bmatrix} 2x_{11} & 4x_{12} \\ 2x_{21} & 2x_{22} \end{bmatrix}, \begin{bmatrix} 2y_{11} & 4y_{12} \\ 2y_{21} & 2y_{22} \end{bmatrix} \right)$$

and all x_{ij} and y_{ij} are in \mathbb{Z} , and where

$$\begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \\ y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \end{bmatrix} = A \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

Next, we try to find necessary and sufficient conditions which such a pair of matrices should satisfy.

Consider

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & -2 & 2 & -1 & 1 & -2 & 2 & -1 \\ 1 & 0 & 2 & -1 & 1 & 0 & 2 & -1 \\ -1 & 2 & 0 & 1 & -1 & 2 & 0 & 1 \\ -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ -1 & 2 & -2 & 1 & 1 & -2 & 2 & -1 \\ -1 & 0 & -2 & 1 & 1 & 0 & 2 & -1 \\ 1 & -2 & 0 & -1 & -1 & 2 & 0 & 1 \end{bmatrix}$$

Simplifying it through elementary row reductions, we obtain an equivalent matrix :

$$A^{-1} \sim \frac{1}{4} \begin{array}{l} r_3 - r_2 \\ r_4 + r_2 \\ r_2 - r_1 \\ r_5 + r_1 \\ r_7 - r_6 \\ r_8 + r_6 \\ r_6 + r_1 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & 2 & -2 & 0 & -2 & 2 & -2 \\ 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 2 & -2 & 2 & 2 & -2 & 2 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

or

$$A^{-1} \sim \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \end{bmatrix}$$

We deduce the following conditions:

$$\begin{aligned} &4|x_{11} + x_{22} + y_{11} + y_{22}, \quad 2|y_{11} + y_{22}, \quad 2|y_{11} + x_{22}, \\ &2|x_{12} + y_{12}, \quad \text{and} \quad 2|x_{21} + y_{21}. \end{aligned} \tag{1.3.1}$$

This is to say that such a pair of matrices with the property (1.3.1) will correspond to an element in $\mathbb{Z}(D_8 \times C_2)(1 - a^2)$. Conversely, it is easy to see any element in $\mathbb{Z}(D_8 \times C_2)(1 - a^2)$ will correspond to such a pair of matrices with the property (1.3.1). As a consequence, we obtain that

$$\begin{aligned} &\mathbb{Z}(D_8 \times C_2)(1 - a^2) \cong \\ &\left(\left\{ \begin{bmatrix} 2x_{11} & 4x_{12} \\ 2x_{21} & 2x_{22} \end{bmatrix}, \begin{bmatrix} 2y_{11} & 4y_{12} \\ 2y_{21} & 2y_{22} \end{bmatrix} \right\} \mid \begin{matrix} 4|x_{11} + x_{22} + y_{11} + y_{22}, & 2|y_{11} + y_{22}, \\ 2|y_{11} + x_{22}, & 2|x_{12} + y_{12}, & 2|x_{21} + y_{21} \end{matrix} \right\} \end{aligned}$$

In addition, notice that such an element v is in $\Delta(D_8 \times C_2)(1 - a^2)$ if and only if $2 \mid \sum_{i=1}^4 (\alpha_i + \beta_i)$, i.e. $2|(y_{11} + 2y_{21})$, or $2|y_{11}$.

($\implies \forall v \in \Delta(D_8 \times C_2)(1 - a^2)$, $v = r(1 - a^2)$ where $r \in \Delta(D_8 \times C_2)$, and we can write $r = r_1 + r_2(1 + a^2)$, where $\text{aug}(r_1) = \sum_{i=1}^4 (\alpha_i + \beta_i)$, $r_1, r_2 \in \mathbb{Z}(D_8 \times C_2)$).

$$0 = \text{aug}(r) = \text{aug}(r_1) + 2\text{aug}(r_2) \implies 2 \mid \text{aug}(r_1) = \sum_{i=1}^4 (\alpha_i + \beta_i).$$

$$\iff \text{If } \sum_{i=1}^4 (\alpha_i + \beta_i) = 2n, \text{ then } v = v - n(1 + a^2)(1 - a^2) \in \Delta(D_8 \times C_2)(1 - a^2).$$

Now the condition (1.3.1) reduces to

$$\begin{aligned} &4 \mid x_{11} + x_{22} + y_{11} + y_{22}, \quad 2 \mid y_{11}, \quad 2 \mid y_{22}, \\ &2 \mid x_{11}, \quad 2 \mid x_{22}, \quad 2 \mid x_{12} + y_{12}, \quad \text{and} \quad 2 \mid x_{21} + y_{21}. \end{aligned} \tag{1.3.2}$$

Rewrite the above pair of matrices in the form

$$\left(\begin{bmatrix} 4w_{11} & 4w_{12} \\ 2w_{21} & 4w_{22} \end{bmatrix}, \begin{bmatrix} 4z_{11} & 4z_{12} \\ 2z_{21} & 4z_{22} \end{bmatrix} \right)$$

where $x_{ii} = 2w_{ii}$, $y_{ii} = 2z_{ii}$, $x_{ij} = w_{ij}$, $y_{ij} = z_{ij}$ $i \neq j$, $i, j = 1, 2$. (1.3.2) becomes

$$2 \mid w_{11} + w_{22} + z_{11} + z_{22}, \quad 2 \mid w_{12} + z_{12}, \quad \text{and} \quad 2 \mid w_{21} + z_{21}. \tag{1.3.3}$$

If $u = 1 + v \in W$, then

$$\begin{bmatrix} 1 + 4w_{11} & 4w_{12} \\ 2w_{21} & 1 + 4w_{22} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 + 4z_{11} & 4z_{12} \\ 2z_{21} & 1 + 4z_{22} \end{bmatrix}$$

are invertible with inverses of the same type. Note that both determinants of the above pair of matrices are 1 since they cannot be -1 , so $2 \mid z_{11} + z_{22}$ and $2 \mid w_{11} + w_{22}$. Thus the first condition in (1.3.3) is redundant.

Conversely we note that if the above pair of matrices are invertible and satisfy (1.3.3), then the inverses are of the same type and also satisfy the same parity conditions. Therefore, the corresponding element is indeed in W .

Now we have proved that

$$W = \{u \in 1 + \Delta(D_8 \times C_2)(1 - a^2) \mid u \in \mathcal{U}(\mathbb{Z}(D_8 \times C_2))\}$$

$$\cong \left\{ \left(\begin{bmatrix} 1+4w_{11} & 4w_{12} \\ 2w_{21} & 1+4w_{22} \end{bmatrix}, \begin{bmatrix} 1+4z_{11} & 4z_{12} \\ 2z_{21} & 1+4z_{22} \end{bmatrix} \right) \mid \begin{array}{l} 2 \mid w_{12} + z_{12} \quad 2 \mid w_{21} + z_{21} \\ w_{ij}, z_{ij} \in \mathbb{Z} \quad \det = 1 \end{array} \right\}$$

1.3.2 Generators of the Normal Complement W

In this subsection, we construct a set of generators for the normal complement W of $D_8 \times C_2$ in $\mathcal{U}_1(\mathbb{Z}(D_8 \times C_2))$. Moreover, we prove that bicyclic units generate W .

Recall that

$$W \cong \left\{ \left(\begin{bmatrix} 1+4w_{11} & 4w_{12} \\ 2w_{21} & 1+4w_{22} \end{bmatrix}, \begin{bmatrix} 1+4z_{11} & 4z_{12} \\ 2z_{21} & 1+4z_{22} \end{bmatrix} \right) \mid \begin{array}{l} 2 \mid w_{12} + z_{12} \quad 2 \mid w_{21} + z_{21} \\ w_{ij}, z_{ij} \in \mathbb{Z} \quad \det = 1 \end{array} \right\}$$

which we denote by H_1 . Let

$$H_2 = \left\{ \left(\begin{bmatrix} 1+4w_{11} & 4w_{12} \\ 2w_{21} & 1+4w_{22} \end{bmatrix}, \begin{bmatrix} 1+4z_{11} & 4z_{12} \\ 2z_{21} & 1+4z_{22} \end{bmatrix} \right) \mid \begin{array}{l} w_{ij}, z_{ij} \in \mathbb{Z} \\ \det = 1 \end{array} \right\}$$

Then H_1 is a subgroup of index 4 in H_2 with a set of coset representatives as follows:

$$\begin{aligned} X_1 &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) & X_2 &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \right) \\ X_3 &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right) & X_4 &= \left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \right) \end{aligned}$$

(It is easy to verify that $X_i X_j^{-1}$ is not in H_1 , for $i \neq j$, and $\forall h \in H_2$, $h X_i^{-1} \in H_1$ for some i , where $i, j = 1, 2, 3, 4$.)

Let's first compute a set of generators for H_2 .

Recall that $\Gamma(2) = \left\{ M = \begin{bmatrix} 1+2a & 2b \\ 2c & 1+2d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}, \det M = 1, M \in PSL(2, \mathbb{Z}) \right\}$ is the principal congruence subgroup of level 2 in $PSL(2, \mathbb{Z})$, so it is free of rank 2 with generators $V_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. Moreover, $\Gamma(2)$ is of index 6 in $PSL(2, \mathbb{Z})$ and

$$\Gamma(4) = \left\{ M = \begin{bmatrix} 1+4a & 4b \\ 4c & 1+4d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}, \det M = 1, M \in PSL(2, \mathbb{Z}) \right\}$$

is of index 24 in $PSL(2, \mathbb{Z})$ (Lemma 1.1.3), so $\Gamma(4)$ is of index 4 in $\Gamma(2)$. Let

$$N = \left\{ M = \begin{bmatrix} 1+4a & 4b \\ 2c & 1+4d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}, \det M = 1, M \in PSL(2, \mathbb{Z}) \right\}.$$

Since $\Gamma(4) \subset N \subset \Gamma(2)$, N is a subgroup of index 2 in $\Gamma(2)$ and a set of coset representatives is as follows:

$$Y_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad Y_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Using the Schreier method, we find that N is generated by

$$h_{1,2,1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad h_{2,1,1} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \quad h_{2,2,2} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

where $h_{i,j,k} = Y_i V_j Y_k^{-1}$, $i, j, k = 1, 2$. Note that we have expanded the notation introduced in Theorem 1.1.2, as it will be useful to keep track of all 3 subscripts.

As a consequence, a set of generators for $H_2 = N \times N$ is as follows:

$$\begin{aligned} Z_1 &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right) & Z_2 &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \right) \\ Z_3 &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix} \right) & Z_4 &= \left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ Z_5 &= \left(\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) & Z_6 &= \left(\begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \end{aligned}$$

Applying the Schreier method again, we find that H_1 is generated by seven elements as follows:

$$\begin{aligned} g_1 = g_{1,3,3} &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 21 & -8 \\ 8 & -3 \end{bmatrix} \right) & g_2 = g_{2,6,4} &= \left(\begin{bmatrix} 21 & -8 \\ 8 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ g_3 = g_{3,4,1} &= \left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right) & g_4 = g_{1,4,3} &= \left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \right) \\ g_5 = g_{2,5,1} &= \left(\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \right) & g_6 = g_{1,5,2} &= \left(\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \right) \\ g_7 = g_{4,1,2} &= \left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 9 & -32 \\ 2 & -7 \end{bmatrix} \right) \end{aligned}$$

where $g_{i,j,k} = X_i Z_j X_k^{-1}$.

1.3.3 Bicyclic Units in $\mathcal{U}(\mathbb{Z}(D_8 \times C_2))$

Next we observe that, up to inverses, $\mathbb{Z}(D_8 \times C_2)$ has only 16 bicyclic units, namely:

$$\begin{aligned}
u_1 &= u_{b,a} = 1 + (1-b)a(1+b) = 1 + (a+ab)(1-a^2) \\
u_2 &= u_{ab,a} = 1 + (1-ab)a(1+ab) = 1 + (a-b)(1-a^2) \\
u_3 &= u_{a^2b,a} = 1 + (1-a^2b)a(1+a^2b) = 1 + (a-ab)(1-a^2) \\
u_4 &= u_{a^3b,a} = 1 + (1-a^3b)a(1+a^3b) = 1 + (a+b)(1-a^2) \\
u_5 &= u_{bc,a} = 1 + (1-bc)a(1+bc) = 1 + (a+abc)(1-a^2) \\
u_6 &= u_{abc,a} = 1 + (1-abc)a(1+abc) = 1 + (a-bc)(1-a^2) \\
u_7 &= u_{a^2bc,a} = 1 + (1-a^2bc)a(1+a^2bc) = 1 + (a-abc)(1-a^2) \\
u_8 &= u_{a^3bc,a} = 1 + (1-a^3bc)a(1+a^3bc) = 1 + (a+bc)(1-a^2) \\
u_9 &= u_{b,ac} = 1 + (1-b)ac(1+b) = 1 + (ac+abc)(1-a^2) \\
u_{10} &= u_{ab,ac} = 1 + (1-ab)ac(1+ab) = 1 + (ac-bc)(1-a^2) \\
u_{11} &= u_{a^2b,ac} = 1 + (1-a^2b)ac(1+a^2b) = 1 + (ac-abc)(1-a^2) \\
u_{12} &= u_{a^3b,ac} = 1 + (1-a^3b)ac(1+a^3b) = 1 + (ac+bc)(1-a^2) \\
u_{13} &= u_{bc,ac} = 1 + (1-bc)ac(1+bc) = 1 + (ab+ac)(1-a^2) \\
u_{14} &= u_{abc,ac} = 1 + (1-abc)ac(1+abc) = 1 + (ac-b)(1-a^2) \\
u_{15} &= u_{a^2bc,ac} = 1 + (1-a^2bc)ac(1+a^2bc) = 1 + (ac-ab)(1-a^2) \\
u_{16} &= u_{a^3bc,ac} = 1 + (1-a^3bc)ac(1+a^3bc) = 1 + (ac+b)(1-a^2)
\end{aligned}$$

Matrices corresponding to these bicyclic units are respectively,

$$u'_1 = \left(\begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix} \right) \quad u'_2 = \left(\begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix} \right)$$

$$u'_3 = \left(\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \right) \quad u'_4 = \left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right)$$

$$u'_5 = \left(\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix} \right) \quad u'_6 = \left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix} \right)$$

$$u'_7 = \left(\begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \right) \quad u'_8 = \left(\begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right)$$

$$u'_9 = \left(\begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix}, \begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix} \right) \quad u'_{10} = \left(\begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix}, \begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix} \right)$$

$$u'_{11} = \left(\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \right) \quad u'_{12} = \left(\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right)$$

$$u'_{13} = \left(\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix} \right) \quad u'_{14} = \left(\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix} \right)$$

$$u'_{15} = \left(\begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \right) \quad u'_{16} = \left(\begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right)$$

But $g_1 = (u'_3)^{-1}(u'_{13})^{-1}(u'_4)^{-1}(u'_{12})^{-1}$, $g_2 = u'_{11}(u'_7)^{-1}u'_{12}(u'_4)^{-1}$, $g_3 = u'_4$, $g_4 = (u'_{12})^{-1}$, $g_5 = (u'_3)^{-1}$, $g_6 = u'_{11}$ and $g_7 = u'_4(u'_{12})^{-1}u'_7(u'_{11})^{-1}(u'_3)^{-1}u'_8u'_3$. We conclude that the seven bicyclic units $u_3, u_4, u_7, u_8, u_{11}, u_{12}$ and u_{13} , generate W . This completes the proof of Theorem 1.3.1.

1.4 Description of $\mathcal{U}(\mathbb{Z}(D_8 \times C_2 \times C_2))$

This section continues the study of the unit group of the integral group ring of the group $G \times C_2$. In particular, it describes the unit group $\mathcal{U}(\mathbb{Z}(D_8 \times C_2 \times C_2))$ of the integral group ring $\mathbb{Z}(D_8 \times C_2 \times C_2)$. Techniques developed in the previous section and the result regarding $\mathcal{U}(\mathbb{Z}(D_8 \times C_2))$ are used to give a matrix representation of a torsion-free normal complement of $D_8 \times C_2 \times C_2$ in $\mathcal{U}_1(D_8 \times C_2 \times C_2)$. More significantly, a set of generators for this normal complement is also constructed, and it turns out that none of the normal complements of $D_8 \times C_2 \times C_2$ can be generated by bicyclic units.

Let $D_8 \times C_2 \times C_2 = \langle a, b, c_1, c_2 \mid a^4 = b^2 = c_1^2 = c_2^2 = 1, ab = ba^{-1}, ac_i = c_i a, \text{ and } bc_i = c_i b, i = 1, 2 \rangle$.

Our main result is as follows:

Theorem 1.4.1. *In $\mathcal{U}_1(\mathbb{Z}(D_8 \times C_2 \times C_2))$, $D_8 \times C_2 \times C_2$ has a torsion-free normal complement $V = \{u = 1 + \alpha(1 - a^2) \mid \alpha \in \Delta(D_8 \times C_2 \times C_2), u \text{ a unit}\}$. V can be represented as a set of four copies of 2×2 matrices with certain parity conditions. Moreover, a set of generators for this torsion-free normal complement is constructed.*

As before, if $A = \langle 1, a^2 \rangle$, then $D_8 \times C_2 \times C_2 / A \cong C_2 \times C_2 \times C_2 \times C_2$, so the first statement is a consequence of Theorem 1.1.1.

1.4.1 A Matrix Presentation of the Normal Complement

If we choose idempotents $f_1 = \frac{1-a^2}{2} \frac{1-c_1}{2} \frac{1-c_2}{2}$, $f_2 = \frac{1-a^2}{2} \frac{1+c_1}{2} \frac{1-c_2}{2}$, $f_3 = \frac{1-a^2}{2} \frac{1-c_1}{2} \frac{1+c_2}{2}$, and $f_4 = \frac{1-a^2}{2} \frac{1+c_1}{2} \frac{1+c_2}{2}$ in the rational group algebra $\mathbb{Q}(D_8 \times C_2 \times C_2)$, then we obtain that

$$\mathbb{Q}(D_8 \times C_2 \times C_2)f_i \cong M_{2 \times 2}(\mathbb{Q}), \text{ where } 1 \leq i \leq 4.$$

Elementary matrix bases for the above are given respectively as follows:

$$\begin{array}{ll} e_{11} = \frac{1+b}{2}f_1 & e_{12} = \frac{ab-a}{2}f_1 \\ e_{21} = \frac{ab+a}{2}f_1 & e_{22} = \frac{1-b}{2}f_1 \\ e'_{11} = \frac{1+b}{2}f_2 & e'_{12} = \frac{ab-a}{2}f_2 \\ e'_{21} = \frac{ab+a}{2}f_2 & e'_{22} = \frac{1-b}{2}f_2 \\ s_{11} = \frac{1+b}{2}f_3 & s_{12} = \frac{ab-a}{2}f_3 \\ s_{21} = \frac{ab+a}{2}f_3 & s_{22} = \frac{1-b}{2}f_3 \\ s'_{11} = \frac{1+b}{2}f_4 & s'_{12} = \frac{ab-a}{2}f_4 \\ s'_{21} = \frac{ab+a}{2}f_4 & s'_{22} = \frac{1-b}{2}f_4 \end{array}$$

Notice that $\Delta(D_8 \times C_2 \times C_2)(1 - a^2) \subseteq \mathbb{Z}(D_8 \times C_2 \times C_2)(1 - a^2) \subseteq \mathbb{Q}(D_8 \times C_2 \times C_2)(1 - a^2) = \mathbb{Q}D_8f_1 \oplus \mathbb{Q}D_8f_2 \oplus \mathbb{Q}D_8f_3 \oplus \mathbb{Q}D_8f_4$.

A typical element in $\mathbb{Z}(D_8 \times C_2 \times C_2)(1 - a^2)$ can be written in the form

$$\begin{aligned} & (\alpha_0 + \alpha_1 a + \alpha_2 b + \alpha_3 ab + (\beta_0 + \beta_1 a + \beta_2 b + \beta_3 ab)c_1)(1 - a^2) + \\ & (\alpha'_0 + \alpha'_1 a + \alpha'_2 b + \alpha'_3 ab + (\beta'_0 + \beta'_1 a + \beta'_2 b + \beta'_3 ab)c_1)c_2(1 - a^2) \end{aligned}$$

or

$$\begin{aligned}
& 2\{[(\alpha_0 - \alpha'_0) - (\beta_0 - \beta'_0)] + [(\alpha_1 - \alpha'_1) - (\beta_1 - \beta'_1)]a + \\
& [(\alpha_2 - \alpha'_2) - (\beta_2 - \beta'_2)]b + [(\alpha_3 - \alpha'_3) - (\beta_3 - \beta'_3)]ab\}f_1 + \\
& 2\{[(\alpha_0 - \alpha'_0) + (\beta_0 - \beta'_0)] + [(\alpha_1 - \alpha'_1) + (\beta_1 - \beta'_1)]a + \\
& [(\alpha_2 - \alpha'_2) + (\beta_2 - \beta'_2)]b + [(\alpha_3 - \alpha'_3) + (\beta_3 - \beta'_3)]ab\}f_2 + \\
& 2\{[(\alpha_0 + \alpha'_0) - (\beta_0 + \beta'_0)] + [(\alpha_1 + \alpha'_1) - (\beta_1 + \beta'_1)]a + \\
& [(\alpha_2 + \alpha'_2) - (\beta_2 + \beta'_2)]b + [(\alpha_3 + \alpha'_3) - (\beta_3 + \beta'_3)]ab\}f_3 + \\
& 2\{[(\alpha_0 + \alpha'_0) + (\beta_0 + \beta'_0)] + [(\alpha_1 + \alpha'_1) + (\beta_1 + \beta'_1)]a + \\
& [(\alpha_2 + \alpha'_2) + (\beta_2 + \beta'_2)]b + [(\alpha_3 + \alpha'_3) + (\beta_3 + \beta'_3)]ab\}f_4
\end{aligned}$$

Since

$f_1 = e_{11} + e_{22}$	$af_1 = e_{21} - e_{12}$
$bf_1 = e_{11} - e_{22}$	$abf_1 = e_{12} + e_{21}$
$f_2 = s_{11} + s_{22}$	$af_2 = s_{21} - s_{12}$
$bf_2 = s_{11} - s_{22}$	$abf_2 = s_{12} + s_{21}$
$f_3 = e'_{11} + e'_{22}$	$af_3 = e'_{21} - e'_{12}$
$bf_3 = e'_{11} - e'_{22}$	$abf_3 = e'_{12} + e'_{21}$
$f_4 = s'_{11} + s'_{22}$	$af_4 = s'_{21} - s'_{12}$
$bf_4 = s'_{11} - s'_{22}$	$abf_4 = s'_{12} + s'_{21}$

such a typical element corresponds to the four copies of 2×2 matrices:

$$\left(\begin{bmatrix} 2[(\alpha_0 - \alpha'_0) - (\beta_0 - \beta'_0)] & 2[(\alpha_3 - \alpha'_3) - (\beta_3 - \beta'_3)] \\ +(\alpha_2 - \alpha'_2) - (\beta_2 - \beta'_2)] & -(\alpha_1 - \alpha'_1) + (\beta_1 - \beta'_1)] \end{bmatrix} \right),$$

$$\left[\begin{array}{cc} 2[(\alpha_0 - \alpha'_0) + (\beta_0 - \beta'_0)] & 2[(\alpha_3 - \alpha'_3) + (\beta_3 - \beta'_3)] \\ +(\alpha_2 - \alpha'_2) + (\beta_2 - \beta'_2)] & -(\alpha_1 - \alpha'_1) - (\beta_1 - \beta'_1)] \\ 2[(\alpha_3 - \alpha'_3) + (\beta_3 - \beta'_3)] & 2[(\alpha_0 - \alpha'_0) + (\beta_0 - \beta'_0)] \\ +(\alpha_1 - \alpha'_1) + (\beta_1 - \beta'_1)] & -(\alpha_2 - \alpha'_2) - (\beta_2 - \beta'_2)] \end{array} \right],$$

$$\left[\begin{array}{cc} 2[(\alpha_0 + \alpha'_0) - (\beta_0 + \beta'_0)] & 2[(\alpha_3 + \alpha'_3) - (\beta_3 + \beta'_3)] \\ +(\alpha_2 + \alpha'_2) - (\beta_2 + \beta'_2)] & -(\alpha_1 + \alpha'_1) + (\beta_1 + \beta'_1)] \\ 2[(\alpha_3 + \alpha'_3) - (\beta_3 + \beta'_3)] & 2[(\alpha_0 + \alpha'_0) - (\beta_0 + \beta'_0)] \\ +(\alpha_1 + \alpha'_1) - (\beta_1 + \beta'_1)] & -(\alpha_2 + \alpha'_2) + (\beta_2 + \beta'_2)] \end{array} \right],$$

$$\left(\begin{bmatrix} 2[(\alpha_0 + \alpha'_0) + (\beta_0 + \beta'_0)] & 2[(\alpha_3 + \alpha'_3) + (\beta_3 + \beta'_3)] \\ +(\alpha_2 + \alpha'_2) + (\beta_2 + \beta'_2)] & -(\alpha_1 + \alpha'_1) - (\beta_1 + \beta'_1)] \\ 2[(\alpha_3 + \alpha'_3) + (\beta_3 + \beta'_3)] & 2[(\alpha_0 + \alpha'_0) + (\beta_0 + \beta'_0)] \\ +(\alpha_1 + \alpha'_1) + (\beta_1 + \beta'_1)] & -(\alpha_2 + \alpha'_2) - (\beta_2 + \beta'_2)] \end{bmatrix} \right)$$

Conjugating each by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, we reduce to

$$\begin{pmatrix}
\begin{bmatrix}
2[(\alpha_0 - \alpha'_0) - (\alpha_1 - \alpha'_1) + (\alpha_2 - \alpha'_2) \\
-(\alpha_3 - \alpha'_3) - (\beta_0 - \beta'_0) + (\beta_1 - \beta'_1) \\
-(\beta_2 - \beta'_2) + (\beta_3 - \beta'_3)] \\
2[(\alpha_1 - \alpha'_1) + (\alpha_3 - \alpha'_3) \\
-(\beta_1 - \beta'_1) - (\beta_3 - \beta'_3)]
\end{bmatrix} &
\begin{bmatrix}
4[-(\alpha_1 - \alpha'_1) + (\alpha_2 - \alpha'_2) \\
+(\beta_1 - \beta'_1) - (\beta_2 - \beta'_2)] \\
2[(\alpha_0 - \alpha'_0) + (\alpha_1 - \alpha'_1) \\
-(\alpha_2 - \alpha'_2) + (\alpha_3 - \alpha'_3) - (\beta_0 - \beta'_0) \\
-(\beta_1 - \beta'_1) + (\beta_2 - \beta'_2) - (\beta_3 - \beta'_3)]
\end{bmatrix}
\end{pmatrix}, \\
\begin{pmatrix}
\begin{bmatrix}
2[(\alpha_0 - \alpha'_0) - (\alpha_1 - \alpha'_1) + (\alpha_2 - \alpha'_2) \\
-(\alpha_3 - \alpha'_3) + (\beta_0 - \beta'_0) - (\beta_1 - \beta'_1) \\
+(\beta_2 - \beta'_2) - (\beta_3 - \beta'_3)] \\
2[(\alpha_1 - \alpha'_1) + (\alpha_3 - \alpha'_3) \\
+(\beta_1 - \beta'_1) + (\beta_3 - \beta'_3)]
\end{bmatrix} &
\begin{bmatrix}
4[-(\alpha_1 - \alpha'_1) + (\alpha_2 - \alpha'_2) \\
-(\beta_1 - \beta'_1) + (\beta_2 - \beta'_2)] \\
2[(\alpha_0 - \alpha'_0) + (\alpha_1 - \alpha'_1) \\
-(\alpha_2 - \alpha'_2) + (\alpha_3 - \alpha'_3) + (\beta_0 - \beta'_0) \\
+(\beta_1 - \beta'_1) - (\beta_2 - \beta'_2) + (\beta_3 - \beta'_3)]
\end{bmatrix}
\end{pmatrix}, \\
\begin{pmatrix}
\begin{bmatrix}
2[(\alpha_0 + \alpha'_0) - (\alpha_1 + \alpha'_1) + (\alpha_2 + \alpha'_2) \\
-(\alpha_3 + \alpha'_3) - (\beta_0 + \beta'_0) + (\beta_1 + \beta'_1) \\
-(\beta_2 + \beta'_2) + (\beta_3 + \beta'_3)] \\
2[(\alpha_1 + \alpha'_1) + (\alpha_3 + \alpha'_3) \\
-(\beta_1 + \beta'_1) - (\beta_3 + \beta'_3)]
\end{bmatrix} &
\begin{bmatrix}
4[-(\alpha_1 + \alpha'_1) + (\alpha_2 + \alpha'_2) \\
+(\beta_1 + \beta'_1) - (\beta_2 + \beta'_2)] \\
2[(\alpha_0 + \alpha'_0) + (\alpha_1 + \alpha'_1) \\
-(\alpha_2 + \alpha'_2) + (\alpha_3 + \alpha'_3) - (\beta_0 + \beta'_0) \\
-(\beta_1 + \beta'_1) + (\beta_2 + \beta'_2) - (\beta_3 + \beta'_3)]
\end{bmatrix}
\end{pmatrix}, \\
\begin{pmatrix}
\begin{bmatrix}
2[(\alpha_0 + \alpha'_0) - (\alpha_1 + \alpha'_1) + (\alpha_2 + \alpha'_2) \\
-(\alpha_3 + \alpha'_3) + (\beta_0 + \beta'_0) - (\beta_1 + \beta'_1) \\
+(\beta_2 + \beta'_2) - (\beta_3 + \beta'_3)] \\
2[(\alpha_1 + \alpha'_1) + (\alpha_3 + \alpha'_3) \\
+(\beta_1 + \beta'_1) + (\beta_3 + \beta'_3)]
\end{bmatrix} &
\begin{bmatrix}
4[-(\alpha_1 + \alpha'_1) + (\alpha_2 + \alpha'_2) \\
-(\beta_1 + \beta'_1) + (\beta_2 + \beta'_2)] \\
2[(\alpha_0 + \alpha'_0) + (\alpha_1 + \alpha'_1) \\
-(\alpha_2 + \alpha'_2) + (\alpha_3 + \alpha'_3) + (\beta_0 + \beta'_0) \\
+(\beta_1 + \beta'_1) - (\beta_2 + \beta'_2) + (\beta_3 + \beta'_3)]
\end{bmatrix}
\end{pmatrix}
\end{pmatrix}.$$

denoted by

$$\left(\begin{bmatrix} 2x_{11} & 4x_{12} \\ 2x_{21} & 2x_{22} \end{bmatrix}, \begin{bmatrix} 2y_{11} & 4y_{12} \\ 2y_{21} & 2y_{22} \end{bmatrix}, \begin{bmatrix} 2x'_{11} & 4x'_{12} \\ 2x'_{21} & 2x'_{22} \end{bmatrix}, \begin{bmatrix} 2y'_{11} & 4y'_{12} \\ 2y'_{21} & 2y'_{22} \end{bmatrix} \right) \quad (1.4.1)$$

where x_{ij} , x'_{ij} , y_{ij} and y'_{ij} are in \mathbb{Z} and

$$\begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \\ y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \\ x'_{11} \\ x'_{12} \\ x'_{21} \\ x'_{22} \\ y'_{11} \\ y'_{12} \\ y'_{21} \\ y'_{22} \end{bmatrix} = A \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \alpha'_0 \\ \alpha'_1 \\ \alpha'_2 \\ \alpha'_3 \\ \beta'_0 \\ \beta'_1 \\ \beta'_2 \\ \beta'_3 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \alpha'_0 \\ \alpha'_1 \\ \alpha'_2 \\ \alpha'_3 \\ \beta'_0 \\ \beta'_1 \\ \beta'_2 \\ \beta'_3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{8}$$

Simplifying by elementary row deductions, we obtain an equivalent matrix:

$$A^{-1} \sim \frac{1}{8}$$

This gives the following 14 conditions:

$$\begin{aligned}
& 8 \mid x_{11} + x_{22} + y_{11} + y_{22} + x'_{11} + x'_{22} + y'_{11} + y'_{22}, \\
& 4 \mid x_{12} + y_{12} + x'_{12} + y'_{12}, \quad 4 \mid x_{21} + y_{21} + x'_{21} + y'_{21}, \\
& 4 \mid x_{22} + y_{22} + x'_{22} + y'_{22}, \quad 4 \mid y_{11} + y_{22} + y'_{11} + y'_{22}, \\
& 4 \mid y_{11} + y_{22} + x'_{11} + x'_{22}, \quad 2 \mid x_{22} + y_{22}, \quad 2 \mid x'_{12} + y'_{12}, \\
& 2 \mid x'_{12} + y_{12}, \quad 2 \mid x'_{21} + y'_{21}, \quad 2 \mid x'_{22} + y'_{22}, \\
& 2 \mid x'_{21} + y_{21}, \quad 2 \mid x'_{22} + y_{22}, \quad 2 \mid x'_{22} + x'_{11}.
\end{aligned} \tag{1.4.2}$$

Similar to the situation of $D_8 \times C_2$, such a typical element is in $\Delta(D_8 \times C_2 \times C_2)(1 - a^2)$ if and only if $2 \mid \sum_{i=0}^3 (\alpha_i + \alpha'_i + \beta_i + \beta'_i)$. Since $\sum_{i=0}^3 (\alpha_i + \alpha'_i + \beta_i + \beta'_i) = y'_{11} + 2y'_{21}$, the above condition reduces to $2 \mid y'_{11}$. Together with (1.4.2), we obtain the following conditions:

$$\begin{aligned}
& 2 \mid x_{ii}, \quad 2 \mid x'_{ii}, \quad 2 \mid y_{ii}, \quad 2 \mid y'_{ii}, \quad 2 \mid x_{ij} + y_{ij}, \quad 2 \mid x'_{ij} + y'_{ij}, \quad 2 \mid x'_{ij} + y_{ij}, \\
& 4 \mid x'_{11} + x'_{22} + y_{11} + y_{22}, \quad 4 \mid y_{11} + y_{22} + y'_{11} + y'_{22}, \\
& 4 \mid x_{22} + y_{22} + x'_{22} + y'_{22}, \quad 4 \mid x_{ij} + x'_{ij} + y_{ij} + y'_{ij}, \\
& 8 \mid x_{11} + x_{22} + y_{11} + y_{22} + x'_{11} + x'_{22} + y'_{11} + y'_{22}, \quad i, j = 1, 2, i \neq j
\end{aligned}$$

Rewrite (1.4.1) as

$$\left(\begin{bmatrix} 4z_{11} & 4z_{12} \\ 2z_{21} & 4z_{22} \end{bmatrix}, \begin{bmatrix} 4w_{11} & 4w_{12} \\ 2w_{21} & 4w_{22} \end{bmatrix}, \begin{bmatrix} 4z'_{11} & 4z'_{12} \\ 2z'_{21} & 4z'_{22} \end{bmatrix}, \begin{bmatrix} 4w'_{11} & 4w'_{12} \\ 2w'_{21} & 4w'_{22} \end{bmatrix} \right)$$

where $x_{ii} = 2z_{ii}$, $x'_{ii} = 2z'_{ii}$, $x_{ij} = z_{ij}$, $x'_{ij} = z'_{ij}$, $y_{ii} = 2w_{ii}$, $y'_{ii} = 2w'_{ii}$, $y_{ij} = w_{ij}$, $y'_{ij} = w'_{ij}$. The above conditions reduce to

$$\begin{aligned}
& 2 \mid z_{12} + w_{12}, \quad 2 \mid z'_{12} + w'_{12}, \quad 2 \mid z_{21} + w_{21}, \quad 2 \mid z'_{21} + w'_{21}, \\
& 2 \mid w_{12} + z'_{12}, \quad 2 \mid w_{21} + z'_{21}, \quad 2 \mid z'_{11} + z'_{22} + w_{11} + w_{22}, \\
& 2 \mid w_{11} + w_{22} + w'_{11} + w'_{22}, \quad 2 \mid z_{22} + w_{22} + z'_{22} + w'_{22}, \\
& 4 \mid z_{12} + w_{12} + z'_{12} + w'_{12}, \quad 4 \mid z_{21} + w_{21} + z'_{21} + w'_{21}, \\
& 4 \mid w_{11} + w_{22} + z_{11} + z_{22} + w'_{11} + w'_{22} + z'_{11} + z'_{22}.
\end{aligned} \tag{1.4.3}$$

Now we claim that in $\mathcal{U}_1(\mathbb{Z}(D_8 \times C_2 \times C_2))$, $D_8 \times C_2 \times C_2$ has a torsion-free normal complement V and

$$V \cong \{(M_1, M_2, M_3, M_4) \mid \det(M_i) = 1, \text{ with the parity conditions (1.4.3*)} \}$$

where

$$M_1 = \begin{bmatrix} 1 + 4z_{11} & 4z_{12} \\ 2z_{21} & 1 + 4z_{22} \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 + 4w_{11} & 4w_{12} \\ 2w_{21} & 1 + 4w_{22} \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 1 + 4z'_{11} & 4z'_{12} \\ 2z'_{21} & 1 + 4z'_{22} \end{bmatrix} \text{ and } M_4 = \begin{bmatrix} 1 + 4w'_{11} & 4w'_{12} \\ 2w'_{21} & 1 + 4w'_{22} \end{bmatrix}$$

and

$$\begin{aligned}
& 2 \mid z_{12} + w_{12}, \quad 2 \mid z'_{12} + w'_{12}, \quad 2 \mid z_{21} + w_{21}, \quad 2 \mid z'_{21} + w'_{21}, \\
& 2 \mid w_{12} + z'_{12}, \quad 2 \mid w_{21} + z'_{21}, \quad 2 \mid z_{22} + w_{22} + z'_{22} + w'_{22}, \\
& 4 \mid z_{12} + w_{12} + z'_{12} + w'_{12}, \quad 4 \mid z_{21} + w_{21} + z'_{21} + w'_{21},
\end{aligned} \tag{1.4.3*}$$

Proof. Let $1 + r \in V$. Then as we showed earlier, it corresponds to a four copies of 2×2 invertible matrices (M_1, M_2, M_3, M_4) with the conditions (1.4.3). We will show that in (1.4.3), the following conditions:

$$2 \mid z'_{11} + z'_{22} + w_{11} + w_{22}, \quad 2 \mid w_{11} + w_{22} + w'_{11} + w'_{22}, \text{ and}$$

$$4 \mid w_{11} + w_{22} + z_{11} + z_{22} + w'_{11} + w'_{22} + z'_{11} + z'_{22}$$

are redundant. Therefore, (1.4.3) reduces to (1.4.3*).

We note that all $\det(M_i) = 1$ for $i = 1, 2, 3, 4$ since $\det \begin{bmatrix} 4l+1 & 4m \\ 4n & 4s+1 \end{bmatrix} \neq -1$ for any $l, m, n, s \in \mathbb{Z}$. It follows that

$$\begin{aligned} z_{11} + z_{22} + 4z_{11}z_{22} - 2z_{12}z_{21} &= 0, & z'_{11} + z'_{22} + 4z'_{11}z'_{22} - 2z'_{12}z'_{21} &= 0, \\ w_{11} + w_{22} + 4w_{11}w_{22} - 2w_{12}w_{21} &= 0, & w'_{11} + w'_{22} + 4w'_{11}w'_{22} - 2w'_{12}w'_{21} &= 0. \end{aligned}$$

Therefore $2 \mid z'_{11} + z'_{22} + w_{11} + w_{22}$ and $2 \mid w_{11} + w_{22} + w'_{11} + w'_{22}$. Furthermore, we have

$$\begin{aligned} & (w_{11} + w_{22} + z_{11} + z_{22} + w'_{11} + w'_{22} + z'_{11} + z'_{22}) - 2(z_{12}z_{21} + z'_{12}z'_{21} + w_{12}w_{21} + w'_{12}w'_{21}) \\ &= 0 \pmod{4} \end{aligned} \tag{1.4.4}$$

Case 1: If $2 \mid z_{21}$, then $2 \mid w_{21}$, $2 \mid z'_{21}$ and $2 \mid w'_{21}$ since $2 \mid z_{21} + w_{21}$, $2 \mid z'_{21} + w_{21}$ and $2 \mid z'_{21} + w'_{21}$. Therefore the second term in (1.4.4) is divisible by 4. As a consequence,

$$w_{11} + w_{22} + z_{11} + z_{22} + w'_{11} + w'_{22} + z'_{11} + z'_{22} \equiv 0 \pmod{4} \tag{1.4.5}$$

Case 2: If $z_{21} \not\equiv 0 \pmod{2}$, then neither are w_{21} , z'_{21} and w'_{21} . Therefore (1.4.4) reduces to

$$(w_{11} + w_{22} + z_{11} + z_{22} + w'_{11} + w'_{22} + z'_{11} + z'_{22}) - 2(z_{12} + w_{12} + z'_{12} + w'_{12}) \equiv 0 \pmod{4}$$

Since both $z_{12} + w_{12}$ and $z'_{12} + w'_{12}$ are divisible by 2, we obtain (1.4.5) again and this finishes the proof of one direction.

Conversely, copy the second part of proof of Theorem 1.3.1 and we are done.

□

1.4.2 Generators for the Normal Complement

In this subsection, we proceed to compute generators for the normal complement V of $D_8 \times C_2 \times C_2$ in $\mathcal{U}_1(\mathbb{Z}(D_8 \times C_2 \times C_2))$.

Recall that $D_8 \times C_2$ has a torsion-free normal complement W in $\mathcal{U}_1(\mathbb{Z}(D_8 \times C_2))$ (Section 1.3.2), where

$$W \cong \left\{ \left(\begin{bmatrix} 1+4w_{11} & 4w_{12} \\ 2w_{21} & 1+4w_{22} \end{bmatrix}, \begin{bmatrix} 1+4z_{11} & 4z_{12} \\ 2z_{21} & 1+4z_{22} \end{bmatrix} \right) \mid \begin{array}{l} 2 \mid w_{12} + z_{12} \quad 2 \mid w_{21} + z_{21} \\ w_{ij}, z_{ij} \in \mathbb{Z} \quad \det = 1 \end{array} \right\}$$

is generated by seven bicyclic units (Section 1.3.3) as follows:

$$\begin{aligned} Y'_1 &= \left(\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \right); & Y'_2 &= \left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right); \\ Y'_3 &= \left(\begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \right); & Y'_4 &= \left(\begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right); \\ Y'_5 &= \left(\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \right); & Y'_6 &= \left(\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right); \\ Y'_7 &= \left(\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix} \right); \end{aligned}$$

Therefore $W \times W$ is generated by 14 elements, namely; Y_1, Y_2, \dots, Y_{14} , where $Y_i = (Y'_i, I')$, and $Y_{i+7} = (I', Y'_i)$, $i = 1, 2, \dots, 7$, and I' is the pair of 2×2 identity matrices. Our normal complement V of $D_8 \times C_2 \times C_2$ is the subgroup of $W \times W$ with additional conditions:

$$\begin{aligned} 2 \mid w_{12} + z'_{12}, \quad 2 \mid w_{21} + z'_{21}, \quad 2 \mid w_{22} + w'_{22} + z_{22} + z'_{22} \\ 4 \mid w_{12} + w'_{12} + z_{12} + z'_{12} \quad 4 \mid w_{21} + w'_{21} + z_{21} + z'_{21} \end{aligned}$$

First we compute the index of this subgroup. Let

- $H_1 = \{A \mid A \in W \times W\} = W \times W$,
- $H_2 = \{A \mid A \in H_1 \text{ and } 2 \mid w_{21} + z'_{21}\}$,
- $H_3 = \{A \mid A \in H_2 \text{ and } 2 \mid w_{12} + z'_{12}\}$,
- $H_4 = \{A \mid A \in H_3 \text{ and } 4 \mid w_{12} + w'_{12} + z_{12} + z'_{12}\}$,
- $H_5 = \{A \mid A \in H_4 \text{ and } 4 \mid w_{21} + w'_{21} + z_{21} + z'_{21}\}$,
- $H_6 = \{A \mid A \in H_5 \text{ and } 2 \mid w_{22} + w'_{22} + z_{22} + z'_{22}\} = V$.

It is easy to check that each H_{i+1} is a subgroup of H_i of index 2 for $i = 1, 2, 3, 4, 5$. Therefore, $V = H_6$ is a subgroup of H_1 of index 32.

Next we use the Schreier method to compute generators step by step. In each step, we compute those for a subgroup and try to reduce the number of them as much as possible.

(i) **Step 1. Calculate Generators for H_2 .**

As we mentioned before, $W \times W = \langle Y_1, Y_2, \dots, Y_{14} \rangle$. A set of coset representatives for H_2 is $X_1 = (I, I, I, I); X_2 = (B, B, I, I)$ where $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

Applying the Schreier method, we obtain a set of generators for H_2 which can be reduced to the following 14:

$$g_1 = g_{1,1,1} = \left(\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right);$$

$$g_2 = g_{1,5,1} = \left(\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right);$$

$$g_3 = g_{1,3,1} = \left(\begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right);$$

$$g_4 = g_{1,6,2}^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right);$$

$$g_5 = g_8 g_{1,4,2} g_8^{-1} = \left(\begin{bmatrix} -3 & -8 \\ 8 & 21 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right);$$

$$g_6 = g_{2,2,1} = \left(\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right);$$

$$g_7 = g_{1,7,1} = \left(\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right);$$

$$g_8 = g_{1,9,2} = \left(\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right);$$

$$g_9 = g_{1,8,1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \right);$$

$$g_{10} = g_{1,12,1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \right);$$

$$g_{11} = g_{1,10,1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \right);$$

$$g_{12} = g_{1,14,1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix} \right);$$

$$g_{13} = g_8^{-1} g_{1,11,2} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -3 & -8 \\ 8 & 21 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right);$$

$$g_{14} = g_8 g_{1,13,2}^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right);$$

where $g_{i,j,k} = X_i Y_j X_k^{-1}, 1 \leq i, k \leq 2, 1 \leq j \leq 14$.

$$\text{Let } B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix}, E = \begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix}$$

$$= D^{-1} B^{-1} C, \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \text{ We rewrite } g_i \text{ as follows:}$$

$$\begin{aligned} g_1 &= (C^{-1}, C^{-1}, I, I); & g_2 &= (C, C^{-1}, I, I); \\ g_3 &= (D, C^{-1}, I, I); & g_4 &= (B^2, I, I, I); \\ g_5 &= (B^{-1} E, I, I, I); & g_6 &= (B^2, B^2, I, I); \\ g_7 &= (C, D, I, I); & g_8 &= (B^{-1}, B^{-1}, B, B); \\ g_9 &= (I, I, C^{-1}, C^{-1}); & g_{10} &= (I, I, C, C^{-1}); \\ g_{11} &= (I, I, D, C^{-1}); & g_{12} &= (I, I, C, D); \\ g_{13} &= (I, I, B^{-1} E, I); & g_{14} &= (I, I, B^2, I). \end{aligned}$$

(ii) Step 2. Calculate Generators for H_3 .

From now on, we confuse notations. We denote generators for a group by $\{Y_i\}$, those for its subgroup by $\{g_k\}$ and a set of coset representatives for the subgroup by $\{X_j\}$.

$H_2 = \langle Y_1, Y_2, \dots, Y_{14} \rangle$ and a set of coset representatives for H_3 is given by $X_1 = (I, I, I, I); X_2 = (I, I, C, C)$. Similar to Step 1, we obtain a set of 15 generators for H_3 as follows:

$$\begin{aligned}
g_1 &= g_{1,1,2}^{-1} = (C, C, C, C); & g_2 &= g_1 g_{1,2,2} = (C^2, I, I, I); \\
g_3 &= g_1^2 g_2^{-1} g_{1,9,2} = (I, C^2, I, I); & g_4 &= g_{2,10,1} = (I, I, C^2, I); \\
g_5 &= g_{1,8,1} g_6 g_7 = (B, B, B, B); & g_6 &= g_{1,4,1} = (B^2, I, I, I); \\
g_7 &= g_{1,6,1} g_6^{-1} = (I, B^2, I, I); & g_8 &= g_{1,14,1} = (I, I, B^2, I); \\
g_9 &= g_{1,3,2} g_1 g_2^{-1} = (DC^{-1}, I, I, I); & g_{10} &= g_{1,7,2} g_1 g_2^{-1} g_3^{-1} = (I, DC^{-1}, I, I); \\
g_{11} &= g_{1,11,2} g_1^2 g_2^{-1} g_3^{-1} g_4^{-1} = (I, I, DC^{-1}, I); & g_{12} &= g_{1,12,2} = (I, I, I, DC^{-1}); \\
g_{13} &= g_{1,5,1} = (B^{-1}E, I, I, I); & g_{14} &= g_{1,13,1} = (I, I, B^{-1}E, I); \\
g_{15} &= g_7 g_6 g_{2,8,2} = (B, B, CBC^{-1}, CBC^{-1}).
\end{aligned}$$

(iii) **Step 3. Calculate Generators for H_4 .**

A set of coset representatives is given by $X_1 = (I, I, I, I); X_2 = (C^2, I, I, I)$.

A set of 16 generators for H_4 is as follows:

$$\begin{aligned}
g_1 &= g_{1,1,1} = (C, C, C, C); & g_2 &= g_{1,3,2} g_4 = (C^2, C^2, I, I); \\
g_3 &= g_4 g_{1,4,2} = (C^2, I, C^2, I); & g_4 &= g_{2,2,1} = (C^4, I, I, I); \\
g_5 &= g_{1,5,1} = (B, B, B, B); & g_6 &= g_{1,6,1} = (B^2, I, I, I); \\
g_7 &= g_{1,7,1} = (I, B^2, I, I); & g_8 &= g_{1,8,1} = (I, I, B^2, I); \\
g_9 &= g_{1,9,2} g_4 = (DC, I, I, I); & g_{10} &= g_{1,10,2} g_2 = (I, DC, I, I); \\
g_{11} &= g_{1,11,2} g_3 = (I, I, DC, I); & g_{12} &= g_{1,12,2} g_1^2 g_2^{-1} g_3^{-1} g_4 = (I, I, I, DC); \\
g_{13} &= g_{1,13,2} g_4 = (B^{-1}EC^2, I, I, I); & g_{14} &= g_{1,14,2} g_3 = (I, I, B^{-1}EC^2, I); \\
g_{15} &= g_{1,15,1} = (B, B, CBC^{-1}, CBC^{-1}); & g_{16} &= g_{2,5,2} = (C^2 BC^{-2}, B, B, B).
\end{aligned}$$

(iv) **Step 4. Calculate Generators for H_5 .**

A set of coset representatives for H_5 : $X_1 = (I, I, I, I)$; $X_2 = (B^2, I, I, I)$.

We obtain the following set of 17 generators for H_5 :

$$\begin{aligned}
 g_1 &= g_{1,1,1} = (C, C, C, C); & g_2 &= g_3 g_{1,3,1} g_4^{-1} = (I, C^2, C^2, I); \\
 g_3 &= g_{1,2,1} = (C^2, C^2, I, I); & g_4 &= g_{1,4,1} = (C^4, I, I, I); \\
 g_5 &= g_{1,5,1} = (B, B, B, B); & g_6 &= g_{1,8,2} g_7 = (I, B^2, B^2, I); \\
 g_7 &= g_{1,7,2} g_8 = (B^2, B^2, I, I); & g_8 &= g_{2,6,1} = (B^4, I, I, I); \\
 g_9 &= g_{1,9,2} g_8 = (DCB^2, I, I, I); & g_{10} &= g_{1,10,2} g_7 = (I, DCB^2, I, I); \\
 g_{11} &= g_{1,11,2} g_8 g_6 g_7^{-1} = (I, I, DCB^2, I); & g_{12} &= g_6^{-1} g_{1,12,2} g_5^2 = (I, I, I, DCB^2); \\
 g_{13} &= g_{1,13,1} = (B^{-1}EC^2, I, I, I); & g_{14} &= g_{1,14,1} = (I, I, B^{-1}EC^2, I); \\
 g_{15} &= g_{2,1,2}^{-1} g_1 = (B^2 C^{-1} B^{-2} C, I, I, I); & g_{16} &= g_{1,16,1} = (C^2 BC^{-2}, B, B, B); \\
 g_{17} &= g_{1,15,1} = (B, B, CBC^{-1}, CBC^{-1}).
 \end{aligned}$$

Remark 1.4.2. *It turns out to be convenient if we replace $g_9, g_{10}, g_{11}, g_{12}$ by*

$$\begin{aligned}
 g'_9 &= (CDB^2, I, I, I) = g_1 g_9 g_8^{-1} g_{15} g_1^{-1} g_8, \quad g'_{10} = (I, CDB^2, I, I), \\
 g'_{11} &= (I, I, CDB^2, I), \quad g'_{12} = (I, I, I, CDB^2).
 \end{aligned}$$

(v) **Step 5. Calculate Generators for $H_6(\cong V)$, the Normal Complement.**

A set of coset representatives is given by $X_1 = (I, I, I, I)$, $X_2 = (I, I, I, CDB^2)$.

A set of 19 generators for our normal complement is as follows:

$$\begin{aligned}
 g_1 &= g_{1,1,1} = (C, C, C, C); & g_2 &= g_{1,3,1} = (C^2, C^2, I, I); \\
 g_3 &= g_{1,2,1} = (I, C^2, C^2, I); & g_4 &= g_{1,4,1} = (C^4, I, I, I); \\
 g_5 &= g_{1,5,1} = (B, B, B, B); & g_6 &= g_{1,7,1} = (B^2, B^2, I, I); \\
 g_7 &= g_{1,6,1} = (I, B^2, B^2, I); & g_8 &= g_{1,8,1} = (B^4, I, I, I); \\
 g_9 &= g_{1,9,2}g_{12} = (D', I, I, D'); & g_{10} &= g_{1,10,2}g_{12} = (I, D', I, D'); \\
 g_{11} &= g_{1,11,2}g_{12} = (I, I, D', D'); & g_{12} &= g_{2,12,1} = (I, I, I, (D')^2); \\
 g_{13} &= g_{1,13,2}g_{12} = (F, I, I, D'); & g_{14} &= g_{1,14,2}g_{12} = (I, I, F, D'); \\
 g_{15} &= g_{1,15,1} = (B^2C^{-1}B^{-2}C, I, I, I); & g_{16} &= g_{1,16,1} = (C^2BC^{-2}, B, B, B); \\
 g_{17} &= g_1^{-1}g_{1,17,1}g_1g_5^{-1} & &= (C^{-1}BCB^{-1}, C^{-1}BCB^{-1}, I, I); \\
 g_{18} &= g_{2,5,2} = (B, B, B, D'B(D')^{-1}); & g_{19} &= g_{2,1,2} = (C, C, C, D'C(D')^{-1}).
 \end{aligned}$$

where $D' = CDB^2$ and $F = B^{-1}D^{-1}B^{-1}C^3 = B^{-1}EC^2$.

With a little modification, we reduce to the following set of generators for the normal complement:

$$\begin{aligned}
s_1 &= (C, C, C, C); & s_2 &= (C^2, C^2, I, I); \\
s_3 &= (I, C^2, C^2, I); & s_4 &= (C^4, I, I, I); \\
s_5 &= (B, B, B, B); & s_6 &= (B^2, B^2, I, I); \\
s_7 &= (I, B^2, B^2, I); & s_8 &= (B^4, I, I, I); \\
s_9 &= (D, C, C, D); & s_{10} &= (C, D, C, D); \\
s_{11} &= (C, C, D, D); & s_{12} &= (I, I, I, (CD)^2); \\
s_{13} &= (F, I, I, D'); & s_{14} &= (I, I, F, D'); \\
s_{15} &= (B^2, C^{-1}B^2C, I, I); & s_{16} &= (C^2, BC^2B^{-1}, I, I); \\
s_{17} &= ((CB)^2, (CB)^2, I, I); & s_{18} &= (B, B, B, CDBD^{-1}C^{-1}); \\
s_{19} &= (C, C, C, DB^2CB^{-2}(D)^{-1}).
\end{aligned}$$

This completes the proof of Theorem 1.4.1.

1.4.3 Bicyclic Units in $\mathcal{U}(\mathbb{Z}(D_8 \times C_2 \times C_2))$

In this subsection, we will calculate all bicyclic units in $\mathcal{U}(\mathbb{Z}(D_8 \times C_2 \times C_2))$ and prove that bicyclic units do not generate the normal complement. There are 64 bicyclic units up to inverses in $\mathcal{U}(\mathbb{Z}(D_8 \times C_2 \times C_2))$. The first 16 bicyclic units are the same as those in $\mathcal{U}(\mathbb{Z}(D_8 \times C_2))$ shown on page 22; in this case, we write c_1 instead of c . The next 16 bicyclic units are obtained by replacing the first subscript α of each $u_{\alpha, \beta}$ by αc_2 . The remaining 32 bicyclic units are produced by replacing the second subscript β of the first 32 bicyclic units obtained by βc_2 .

Matrices corresponding to these bicyclic units are respectively,

$$\begin{aligned}
u'_1 &= (D, D, D, D); & u'_2 &= (E, E, E, E); \\
u'_3 &= (C^{-1}, C^{-1}, C^{-1}, C^{-1}); & u'_4 &= (B, B, B, B); \\
u'_5 &= (C^{-1}, D, C^{-1}, D); & u'_6 &= (B, E, B, E); \\
u'_7 &= (D, C^{-1}, D, C^{-1}); & u'_8 &= (E, B, E, B); \\
u'_9 &= (D^{-1}, D, D^{-1}, D); & u'_{10} &= (E^{-1}, E, E^{-1}, E); \\
u'_{11} &= (C, C^{-1}, C, C^{-1}); & u'_{12} &= (B^{-1}, B, B^{-1}, B); \\
u'_{13} &= (C, D, C, D); & u'_{14} &= (B^{-1}, E, B^{-1}, E); \\
u'_{15} &= (D^{-1}, C^{-1}, D^{-1}, C^{-1}); & u'_{16} &= (E^{-1}, B, E^{-1}, B); \\
u'_{17} &= (C^{-1}, C^{-1}, D, D); & u'_{18} &= (B, B, E, E); \\
u'_{19} &= (D, D, C^{-1}, C^{-1}); & u'_{20} &= (E, E, B, B); \\
u'_{21} &= (D, C^{-1}, C^{-1}, D); & u'_{22} &= (E, B, B, E); \\
u'_{23} &= (C^{-1}, D, D, C^{-1}); & u'_{24} &= (B, E, E, B); \\
u'_{25} &= (C, C^{-1}, D^{-1}, D); & u'_{26} &= (B^{-1}, B, E^{-1}, E); \\
u'_{27} &= (D^{-1}, D, C, C^{-1}); & u'_{28} &= (E^{-1}, E, B^{-1}, B); \\
u'_{29} &= (D^{-1}, C^{-1}, C, D); & u'_{30} &= (E^{-1}, B, B^{-1}, E); \\
u'_{31} &= (C, D, D^{-1}, C^{-1}); & u'_{32} &= (B^{-1}, E, E^{-1}, B);
\end{aligned}$$

$$\begin{aligned}
u'_{33} &= (D^{-1}, D^{-1}, D, D); & u'_{34} &= (E^{-1}, E^{-1}, E, E); \\
u'_{35} &= (C, C, C^{-1}, C^{-1}); & u'_{36} &= (B^{-1}, B^{-1}, B, B); \\
u'_{37} &= (C, D^{-1}, C^{-1}, D); & u'_{38} &= (B^{-1}, E^{-1}, B, E); \\
u'_{39} &= (D^{-1}, C, D, C^{-1}); & u'_{40} &= (E^{-1}, B^{-1}, E, B); \\
u'_{41} &= (D, D^{-1}, D^{-1}, D); & u'_{42} &= (E, E^{-1}, E^{-1}, E); \\
u'_{43} &= (C^{-1}, C, C, C^{-1}); & u'_{44} &= (B, B^{-1}, B^{-1}, B); \\
u'_{45} &= (C^{-1}, D^{-1}, C, D); & u'_{46} &= (B, E^{-1}, B^{-1}, E); \\
u'_{47} &= (D, C, D^{-1}, C^{-1}); & u'_{48} &= (E, B^{-1}, E^{-1}, B); \\
u'_{49} &= (C, C, D, D); & u'_{50} &= (B^{-1}, B^{-1}, E, E); \\
u'_{51} &= (D^{-1}, D^{-1}, C^{-1}, C^{-1}); & u'_{52} &= (E^{-1}, E^{-1}, B, B); \\
u'_{53} &= (D^{-1}, C, C^{-1}, D); & u'_{54} &= (E^{-1}, B^{-1}, B, E); \\
u'_{55} &= (C, D^{-1}, D, C^{-1}); & u'_{56} &= (B^{-1}, E^{-1}, E, B); \\
u'_{57} &= (C^{-1}, C, D^{-1}, D); & u'_{58} &= (B, B^{-1}, E^{-1}, E); \\
u'_{59} &= (D, D^{-1}, C, C^{-1}); & u'_{60} &= (E, E^{-1}, B^{-1}, B); \\
u'_{61} &= (D, C, C, D); & u'_{62} &= (E, B^{-1}, B^{-1}, E); \\
u'_{63} &= (C^{-1}, D^{-1}, D^{-1}, C^{-1}); & u'_{64} &= (B, E^{-1}, E^{-1}, B).
\end{aligned}$$

We note that each of the following generators: $s_1, s_2, \dots, s_{12}, s_{17}$, is a product of some bicyclic units. Explicitly, $s_1 = (u'_3)^{-1}$, $s_2 = (u'_3)^{-1}u'_{35}$, $s_3 = (u'_3)^{-1}u'_{43}$, $s_4 = (u'_3)^{-1}u'_{11}(u'_3)^{-1}u'_{35}(u'_{43})^{-1}u'_3$, $s_5 = u'_4$, $s_6 = u'_4(u'_{36})^{-1}$, $s_7 = u'_4(u'_{44})^{-1}$, $s_8 = u'_4(u'_{12})^{-1}u'_4(u'_{36})^{-1}u'_{44}(u'_4)^{-1}$, $s_9 = u'_{61}$, $s_{10} = u'_{13}$, $s_{11} = u'_{49}$, $s_{12} = (u'_{31})^{-1}u'_{25}(u'_3)^{-1}u'_{13}(u'_{43})^{-1}u'_{35}(u'_3)^{-1}u'_{43}(u'_{35})^{-1}u'_3(u'_{11})^{-1}u'_3$, $s_{17} = u'_{49}u'_{18}u'_{35}u'_4$.

Next we will prove that s_{13} is not a product of bicyclic units. Since

$$B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix},$$

are free generators for a subgroup K of index 2 in $\Gamma(2)$ (see [29]), we can define a mapping as follows:

$$f : K \times K \times K \times K \rightarrow D_8 \times D_8 \times D_8 \times D_8,$$

where $f = \bigoplus_{i=1}^4 f_i$, $f_i : K \rightarrow D_8$, $f_i(B) = a$, $f_i(C) = f_i(D) = b$, $i = 1, 2, 3, 4$. Note that $f_i(D') = a^2$, $f_i(E) = a$, $f_i(F) = 1$. So $f(s_{13}) = (1, 1, 1, a^2)$ and images of all the bicyclic units up to inverses are reduced to $\gamma_1 = f(u'_1) = (b, b, b, b)$; $\gamma_2 = f(u'_2) = (a, a, a, a)$; $\gamma_3 = f(u'_{10}) = (a^{-1}, a, a^{-1}, a)$; $\gamma_4 = f(u'_{34}) = (a^{-1}, a^{-1}, a, a)$; and $\gamma_5 = f(u'_{42}) = (a, a^{-1}, a^{-1}, a)$. Since $\gamma_1 \gamma_i = \gamma_i^{-1} \gamma_1$, and $\gamma_1^2 = (1, 1, 1, 1)$, we have that if $f(s_{13})$ is a product of $\gamma_i^{\pm 1}$, then $f(s_{13})$ is a product of $\gamma_i^{\pm 1}$ with $i > 1$. Let

$p : \langle a \rangle \times \langle a \rangle \times \langle a \rangle \times \langle a \rangle \rightarrow \langle a \rangle$ be defined by $p(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4$. Then p is a homomorphism. Since $p(f(s_{13})) = a^2$ and $p(\prod_{i>1} \gamma_i^m) = \prod_{i>1} p(\gamma_i)^m = 1$, s_{13} cannot be a product of bicyclic units. We have proved the following proposition:

Proposition 1.4.3. *The normal complement V of $D_8 \times C_2 \times C_2$ cannot be generated by bicyclic units.*

Since all of the bicyclic units are situated inside this normal complement V , we even have a stronger result:

Corollary 1.4.4. *None of the normal complements of $D_8 \times C_2 \times C_2$ can be generated by bicyclic units.*

1.5 Units in Other Integral Group Rings

The procedure developed in previous sections can be applied to other integral group rings. We give two examples as follows:

Theorem 1.5.1. *In $\mathcal{U}_1(\mathbb{Z}(D_6 \times C_2))$, $D_6 \times C_2$ has a torsion-free normal complement $W = \{u = 1 + \alpha(1 - a^2) | \alpha \in \Delta(D_6 \times C_2), u \text{ a unit}\}$ which is generated by 7 bicyclic units. More explicitly,*

$$W \cong \left\{ \left(\begin{bmatrix} 1 + 3w_{11} & 3w_{12} \\ 3w_{21} & 1 + 3w_{22} \end{bmatrix}, \begin{bmatrix} 1 + 3z_{11} & 3z_{12} \\ z_{21} & 1 + 3z_{22} \end{bmatrix} \right) \mid \begin{array}{l} 2 \mid w_{ij} + z_{ij} \quad i, j = 1, 2. \\ w_{ij}, z_{ij} \in \mathbb{Z} \quad \det = 1 \end{array} \right\}$$

The 7 bicyclic units which generate the normal complement are $u_{ba,ac}$, $u_{ba^2c,ac}$, u_{bac,a^2} , u_{ba^2,a^2} , u_{bc,a^2} , u_{ba,a^2} and u_{ba^2c,a^2} , where $C_2 = \langle c \rangle$.

Theorem 1.5.2. *In $\mathcal{U}_1(\mathbb{Z}D_{10})$, D_{10} has a torsion-free normal complement $W = \{u = 1 + \alpha(1 - a) | \alpha \in \Delta(D_{10}), u \text{ a unit}\}$. More explicitly,*

$$W \cong \left\{ \left[\begin{array}{cc} 1 + w_{11} + z_{11}X & w_{12} + z_{12}X \\ w_{21} + z_{21}X & 1 + w_{22} + z_{22}X \end{array} \right] \mid \begin{array}{l} 5 \mid w_{ij} + 2z_{ij} \quad i, j = 1, 2. \\ w_{ij}, z_{ij} \in \mathbb{Z} \quad \det = 1 \end{array} \right\}$$

where $X^2 + X = 1$.

We have also obtained a result similar to Theorem 1.5.2 for $\mathbb{Z}D_{14}$.

Chapter 2

Unitary and Generalized Unitary Units in Integral Group Rings

In the first chapter, we showed that none of the normal complements of $D_8 \times C_2 \times C_2$ in $\mathcal{U}_1(\mathbb{Z}(D_8 \times C_2 \times C_2))$ is generated by bicyclic units (Corollary 1.4.4), although $D_8 \times C_2$ has a torsion-free normal complement generated by bicyclic units (Theorem 1.3.1). Therefore, the transition of bicyclic generators of a normal complement of G through the operation $G \times C_2$ fails. In this chapter, we will discuss another kind of units called unitary units and show that if in $\mathcal{U}_1(\mathbb{Z}G)$, G has a normal complement generated by unitary units, then this is also true for $G \times C_2$. We also discuss when the unitary units generate a subgroup of finite index in the unit group $\mathcal{U}(\mathbb{Z}G)$ (Section 2.1). Then in section 2.2, we introduce and characterize generalized unitary units. It turns out that these units form a subgroup which is exactly the normalizer of the subgroup of all unitary units. In subsection 2.2.1, we also show that the normalizer of the subgroup of generalized unitary units is equal to itself when G is periodic (Theorem 2.2.4). Subsection 2.2.2 discusses conditions for the unit group being generalized unitary. This is first studied by Bovdi and Sehgal (Theorem 2.2.15) and

by showing their first condition is also sufficient, we obtain necessary and sufficient conditions for the unit group being generalized unitary when G is periodic (Theorem 2.2.18). The characterization of all bicyclic units being nontrivial and generalized unitary is also given. Subsection 2.2.3 discusses conditions for all generalized unitary units being unitary. Subsection 2.2.4 studies an analog of the normalizer conjecture, and also examines the relationship between generalized unitary units in $\mathbb{Z}G$ and $\mathbb{Z}(G \times C_2)$.

2.1 Unitary Units in Integral Group Rings

Let $\mathbb{Z}G$ be the integral group ring of an arbitrary group G and let $f : G \longrightarrow \mathcal{U}(\mathbb{Z}) = \{\pm 1\}$ be any group homomorphism, called an orientation homomorphism of the group G [10]. For each $x = \sum_{g \in G} \alpha_g g \in \mathbb{Z}G$, put $x^f = \sum_{g \in G} \alpha_g f(g)g^{-1}$. Then the mapping $x \mapsto x^f$ is an antiautomorphism of the ring $\mathbb{Z}G$ and is called the involution generated by the homomorphism f . In particular, if f is trivial, x^f coincides with the standard x^* and the above mapping is just the standard involution.

Let $\mathcal{U}(\mathbb{Z}G)$ be the group of units of $\mathbb{Z}G$. Then $u \in \mathcal{U}(\mathbb{Z}G)$ is called f -unitary if $u^{-1} = u^f$ or $u^{-1} = -u^f$ i.e. $uu^f = \pm 1$. It is clear that all f -unitary elements of $\mathcal{U}(\mathbb{Z}G)$ form a subgroup $\mathcal{U}_f(\mathbb{Z}G)$ (or \mathcal{U}_f) containing $G \times \mathcal{U}(\mathbb{Z})$ and we refer to $\mathcal{U}_f(\mathbb{Z}G)$ as the f -unitary subgroup of $\mathcal{U}(\mathbb{Z}G)$. Interest in the group $\mathcal{U}_f(\mathbb{Z}G)$ arose in algebraic topology and unitary K -theory [43]. Novikov posed the problem of investigation of the structure of this group.

If f is trivial, then $\mathcal{U}_f(\mathbb{Z}G) = G \times \mathcal{U}(\mathbb{Z}) = \pm G$, so in that case $\mathcal{U}(\mathbb{Z}G) = \mathcal{U}_f(\mathbb{Z}G)$ if and only if all of the units are trivial. This is characterized by Higman's Theorem Theorem 1.0.1. Hence there is interest in the structure of this group when f is a

nontrivial orientation homomorphism.

In the 1980's, Bovdi first described such a group (see [10] for details). When G is Abelian, Bovdi [13] and also Hoechsmann and Sehgal [23] have given a linearly independent set of generators for a torsion free subgroup of finite index in $\mathcal{U}_f(\mathbb{Z}G)$ and computed the rank of $\mathcal{U}_f(\mathbb{Z}G)$. If $\mathcal{U}_f(\mathbb{Z}G) = \mathcal{U}(\mathbb{Z}G)$, then $\mathcal{U}(\mathbb{Z}G)$ is said to be f -unitary. Bovdi [10] obtained necessary conditions for $\mathcal{U}(\mathbb{Z}G) = \mathcal{U}_f(\mathbb{Z}G)$ and, moreover, proved that most cases of these conditions are also sufficient. Later Bovdi and Sehgal [15] discussed when all bicyclic units are unitary and generate a nontrivial subgroup. Recently, in [14] they continued the study of the unitary subgroup and characterized when such a subgroup is a normal subgroup of the unit group. Most recently, Parmenter [45] discussed the unitary units in integral group rings of groups of order 16.

In this section, we will continue the investigation initiated in Chapter 1 and establish a relationship between unitary units in $\mathbb{Z}G$ and $\mathbb{Z}(G \times C_2)$.

The following notations will be used throughout:

$\mathcal{U}_f(\mathbb{Z}G) = \{ \text{all of the unitary units of } \mathbb{Z}G \}$. Sometimes we will shorten this as \mathcal{U}_f .

$\mathcal{V}_f = \{ \text{all of the unitary units such that } u^f = u^{-1} \}$. It is easy to check \mathcal{V}_f is a normal subgroup of index ≤ 2 in \mathcal{U}_f and \mathcal{V}_f is proper if f is not trivial.

$\mathcal{C} = \{ \text{all of the central units} \}$.

$\mathcal{B}_1(\mathbb{Z}G)$, the subgroup generated by all Bass cyclic units.

$\mathcal{B}_2(\mathbb{Z}G)$, the subgroup generated by all bicyclic units.

If $f : G \rightarrow \mathcal{U}(\mathbb{Z})$ is an orientation homomorphism of G , we extend it to an orientation homomorphism f_1 of $G \times C_2$ by $f_1(gc^i) = f(g)$. When we discuss unitary

units in both group rings $\mathbb{Z}G$ and $\mathbb{Z}(G \times C_2)$, we always mean the f -unitary units in $\mathbb{Z}G$ and the f_1 -unitary units in $\mathbb{Z}(G \times C_2)$.

We have the following theorem:

Theorem 2.1.1. *For an arbitrary group G , $\mathcal{U}(\mathbb{Z}G) = \mathcal{U}_f(\mathbb{Z}G)$ implies $\mathcal{U}(\mathbb{Z}(G \times C_2)) = \mathcal{U}_{f_1}(\mathbb{Z}(G \times C_2))$.*

First we establish several preliminary results.

Proposition 2.1.2. *If G is an arbitrary group, then $\mathcal{U}(\mathbb{Z}(G \times C_2))$ is a semi direct product of K and D , i.e. $\mathcal{U}(\mathbb{Z}(G \times C_2)) = K \rtimes D$,*

where $K = \{u = 1 + \alpha(1 - c) \mid \alpha \in \mathbb{Z}G \text{ and } u \in \mathcal{U}(\mathbb{Z}(G \times C_2))\}$ and $D = \mathcal{U}(\mathbb{Z}G) \subset \mathcal{U}(\mathbb{Z}(G \times C_2))$. Moreover, $1 + \alpha(1 - c)$ is in $\mathcal{U}(\mathbb{Z}(G \times C_2))$ if and only if $1 + 2\alpha$ is in $\mathcal{U}(\mathbb{Z}G)$.

Proof. Recall that in section 1.2, we introduced a homomorphism $f_1 : \mathbb{Z}(G \times C_2) \longrightarrow \mathbb{Z}G$. This implies an exact sequence:

$$1 \longrightarrow K \longrightarrow \mathcal{U}(\mathbb{Z}(G \times C_2)) \longrightarrow \mathcal{U}(\mathbb{Z}G) \longrightarrow 1$$

which proves the first statement.

If $u = 1 + \alpha(1 - c)$ is a unit in K , then $u^{-1} = 1 + \beta(1 - c)$ for some $\beta \in \mathbb{Z}G$. $(1 + \alpha(1 - c))(1 + \beta(1 - c)) = 1$, if and only if $(\alpha + \beta + 2\alpha\beta)(1 - c) = 0$, if and only if $(\alpha + \beta + 2\alpha\beta) = 0$, if and only if $(1 + 2\alpha)(1 + 2\beta) = 1$. This finishes the proof of the proposition. \square

Proposition 2.1.3. *$K = \{u = 1 + \alpha(1 - c) \mid \alpha \in \mathbb{Z}G \text{ and } u \in \mathcal{U}(\mathbb{Z}(G \times C_2))\}$ is isomorphic to $H = \{u = 1 + 2\alpha \mid u \in \mathcal{U}(\mathbb{Z}G)\}$ via the map $1 + \alpha(1 - c) \longrightarrow 1 + 2\alpha$.*

Proof. Using Proposition 2.1.2, it is easy to see that the map $h : K \longrightarrow H$ defined by $h(1 + \alpha(1 - c)) = 1 + 2\alpha$ is a bijection. We only need to show that h is a homomorphism. To this end, let $u_1 = 1 + \alpha_1(1 - c)$, $u_2 = 1 + \alpha_2(1 - c) \in K$. Then

$$\begin{aligned} h(u_1 u_2) &= h(1 + \alpha_1(1 - c) + \alpha_2(1 - c) + \alpha_1(1 - c)\alpha_2(1 - c)) \\ &= 1 + 2(\alpha_1 + \alpha_2 + 2\alpha_1\alpha_2) \end{aligned}$$

On the other hand,

$$\begin{aligned} h(u_1)h(u_2) &= (1 + 2\alpha_1)(1 + 2\alpha_2) \\ &= 1 + 2(\alpha_1 + \alpha_2 + 2\alpha_1\alpha_2) \end{aligned}$$

Therefore, $h(u_1 u_2) = h(u_1)h(u_2)$ and h is an isomorphism. \square

Proposition 2.1.4. *If $1 + 2\alpha$ is a unitary unit, then it must be one of the first class unitary units, i.e. $(1 + 2\alpha)^f = (1 + 2\alpha)^{-1}$.*

Proof. Suppose $(1 + 2\alpha)^f = -(1 + 2\alpha)^{-1}$, then $\text{aug}(2(1 + \alpha^f + \alpha)) = \text{aug}((1 + 2\alpha)^f) + \text{aug}(1 + 2\alpha) = \text{aug}(-(1 + 2\alpha)^{-1}) + \text{aug}(1 + 2\alpha) = 0$, where aug is the augmentation map. Therefore $\text{aug}(\alpha + \alpha^f) = -1$, but $\text{aug}(\alpha + \alpha^f) = \text{aug}(\sum a_g g + a_g f(g)g^{-1}) = \sum a_g(1 + f(g))$. Since $f(g) = \pm 1, 2 \mid (f(g) + 1)$, hence $2 \mid \sum a_g(1 + f(g)) = \text{aug}(\alpha + \alpha^f) = -1$. This contradiction leads to $(1 + 2\alpha)^f = (1 + 2\alpha)^{-1}$. \square

Now we can prove that the isomorphism h in Proposition 2.1.3 induces an isomorphism between the f_1 -unitary units of K and the f -unitary units of H .

Proposition 2.1.5. *$1 + 2\alpha$ is a unitary unit in $\mathcal{U}(\mathbb{Z}G)$ if and only if, $1 + \alpha(1 - c)$ is a unitary unit in $\mathcal{U}(\mathbb{Z}(G \times C_2))$.*

Proof. Suppose $(1 + 2\alpha)$ is a unitary unit. Then by Proposition 2.1.4, $(1 + 2\alpha)^f = (1 + 2\alpha)^{-1}$. Hence $(1 + 2\alpha)(1 + 2\alpha)^f = 1$, forcing $\alpha + \alpha^f + 2\alpha\alpha^f = 0$. Now $(1 + \alpha(1 - c))^{f_1} = 1 + (1 - c)^{f_1}\alpha^{f_1} = 1 + \alpha^f(1 - c)$. Hence $(1 + \alpha(1 - c))^{f_1}(1 + \alpha(1 - c)) = [1 + \alpha^f(1 - c)][1 + \alpha(1 - c)] = 1 + (\alpha + \alpha^f + 2\alpha\alpha^f)(1 - c) = 1$. This finishes one direction.

Conversely, by the above expressions we know that $\pm 1 = 1 + (\alpha + \alpha^f + 2\alpha\alpha^f)(1 - c)$. This forces $(\alpha + \alpha^f + 2\alpha\alpha^f) = 0$, so $(1 + 2\alpha)^f = (1 + 2\alpha)^{-1}$. \square

The proof of Theorem 2.1.1 follows immediately by the above propositions.

Proof. By Proposition 2.1.2, $\mathcal{U}(\mathbb{Z}(G \times C_2)) = K \rtimes D$ where $D = \mathcal{U}(\mathbb{Z}G) = \mathcal{U}_f(\mathbb{Z}G) \subset \mathcal{U}_{f_1}(\mathbb{Z}(G \times C_2))$. Since $H = \{u = 1 + 2\alpha | u \in \mathcal{U}(\mathbb{Z}G)\} \subset \mathcal{U}_f(\mathbb{Z}G)$, $K \subset \mathcal{U}_{f_1}(\mathbb{Z}(G \times C_2))$ by Proposition 2.1.5. Therefore, $\mathcal{U}(\mathbb{Z}(G \times C_2)) = \mathcal{U}_{f_1}(\mathbb{Z}(G \times C_2))$. \square

A natural question to ask now is whether Theorem 2.1.1 can be generalized to a result concerning finite index. We prove next that such an investigation yields nothing new.

Theorem 2.1.6. *For an arbitrary group G , the following conditions are equivalent:*

- (1) $[\mathcal{U} : \mathcal{U}_f] < \infty$;
- (2) $\forall u \in \mathcal{U}, \exists n, \ni u^n \in \mathcal{U}_f$, where n depends on u ;
- (3) $\forall u \in \mathcal{U}, \exists n, \ni (uu^f)^n \in \mathcal{U}_f$;
- (4) $\mathcal{U} = \mathcal{U}_f$.

Proof. (1) \implies (2) \implies (3) and (4) \implies (1) are obvious.

We only need to prove (3) \implies (4). Suppose $\forall u \in \mathcal{U}$, there exists n such that $(uu^f)^n \in \mathcal{U}_f$. Thus $(uu^f)^{2n} = (uu^f)^n(uu^f)^n = 1$ (since $\text{aug}(uu^f)^{2n} = 1$). There-

fore uu^f is a torsion unit. Let $uu^f = z_0 + \sum_{g_i \neq 1} z_i g_i$. We will prove that $z_0 \neq 0$, forcing $uu^f = z_0$ (by Sehgal [57], p. 45, Corollary 1.3) and this finishes the proof.

Let $u = \sum a_i g_i$, so $u^f = \sum a_i f(g_i) g_i^{-1}$. Then

$$\begin{aligned} uu^f &= z_0 + \sum_{i \neq j} a_i a_j g_i g_j^{-1} f(g_j) \\ &= z_0 + \sum_{i < j} (a_i a_j f(g_j) g_i g_j^{-1} + a_j a_i f(g_i) g_j g_i^{-1}) \end{aligned}$$

Therefore, $\pm 1 = \text{aug}(uu^f) = z_0 + \sum_{i < j} a_i a_j (f(g_i) + f(g_j)) \equiv z_0 \pmod{2}$, thus $z_0 \neq 0$ as desired. \square

Example 2.1.7. *As an illustrative example, we study here $\mathcal{U}(\mathbb{Z}D_{16})$.*

We will show that $\mathcal{B}_2(\mathbb{Z}D_{16})$ is a subgroup of infinite index in $\mathcal{U}(\mathbb{Z}D_{16})$. To see this, first let us consider an orientation homomorphism f , defined by $f(a) = 1, f(b) = -1$, on the group D_{16} . We claim $\mathcal{U}_f(\mathbb{Z}D_{16}) \neq \mathcal{U}(\mathbb{Z}D_{16})$. In fact, let $u = 1 - (1 + a^3 + b - ab)(1 - a^4)$, thus $u^f = 1 + (-1 + a + b - ab)(1 - a^4)$. Therefore $uu^f = 1 + (-2 + a - a^3)(1 - a^4)$, so u is not an f -unitary unit (note that $(uu^f)^{-1} = 1 - (2 + a - a^3)(1 - a^4)$). Then we note that $\mathcal{B}_2(\mathbb{Z}D_{16}) \subset \mathcal{U}_f(\mathbb{Z}D_{16})$ by Parmenter ([45], Corollary 5). If $\mathcal{B}_2(\mathbb{Z}D_{16})$ is a subgroup of finite index in $\mathcal{U}(\mathbb{Z}D_{16})$, so is $\mathcal{U}_f(\mathbb{Z}D_{16})$. Therefore $\mathcal{U}_f(\mathbb{Z}D_{16}) = \mathcal{U}(\mathbb{Z}D_{16})$ by Theorem 2.1.6. This contradiction gives the result.

We note that there are two other nontrivial orientation homomorphisms in D_{16} : f_1 , defined by $f_1(a) = -1, f_1(b) = 1$; and f_2 , defined by $f_2(a) = -1, f_2(b) = -1$. Furthermore, $\mathcal{U}_{f_1}(\mathbb{Z}D_{16}) \neq \mathcal{U}(\mathbb{Z}D_{16})$ and $\mathcal{U}_{f_2}(\mathbb{Z}D_{16}) \neq \mathcal{U}(\mathbb{Z}D_{16})$.

For completeness, we recall a result proved primarily by Bovdi [10] (see also Bovdi and Sehgal [14]) describing when $\mathcal{U} = \mathcal{U}_f$.

Theorem 2.1.8. *Let $f : G \rightarrow \mathcal{U}(\mathbb{Z})$ be a nontrivial homomorphism with kernel A and let $\mathcal{U}(\mathbb{Z}G)$ be f -unitary. Then G contains an element b such that $G = \langle A, b \rangle$ and one of the following conditions is satisfied:*

(1) *A is an abelian group, the exponent of its torsion subgroup divides 4 or 6, the order of the element b divides 4 and $bab^{-1} = a^{-1}$ for all $a \in A$;*

(2) *A is a Hamiltonian 2-group and G is the semidirect product of A and $\langle b \mid b^2 = 1 \rangle$, and every subgroup of A is normal in G ;*

(3) *A is a Hamiltonian 2-group and G is a semidirect product of a Hamiltonian 2-group and the cyclic group $\langle b \rangle$ of order 4;*

(4) *$t(A)$ is a central subgroup of G , $t(A)$ is the direct product of $\langle b^2 \mid b^8 = 1 \rangle$ and a group o and $bab^{-1} = a^{-1}b^{4i}$ for all $a \in A$;*

(5) *$t(G)$ is a subgroup, every subgroup of $t(G)$ is normal in G and $t(G)$ satisfies one of the following conditions:*

(5.1) *either $t(G)$ is abelian with exponent a divisor of 4 or 6 or $t(G)$ is a Hamiltonian 2-group;*

(5.2) *$t(G)$ is the direct product of a cyclic group of order 4 and an abelian group whose exponent divides 6;*

(5.3) *$t(G)$ is the direct product of a cyclic group of order 8 and an abelian group whose exponent divides 4.*

Conversely, let G satisfy one of the conditions (1) -(3) or (5) with the further condition that $G/t(G)$ is a right ordered group. If $f : G \rightarrow \mathcal{U}(\mathbb{Z})$ is a homomorphism with kernel A , then $\mathcal{U}(\mathbb{Z}G)$ is f -unitary.

We note that the sufficiency of case (4) still remains open. However, when G is periodic, case (4) becomes a special case of case (5.3), so the conditions of Theorem 2.1.8 become sufficient as well as necessary.

2.2 Generalized Unitary Units

In this section, we first introduce a new kind of units which generalize the unitary units. We prove that these generalized unitary units form a subgroup $\mathcal{U}_{g,f}$ which happens to be the normalizer of \mathcal{U}_f in \mathcal{U} . Then we study the second normalizer of \mathcal{U}_f . We also characterize when $\mathcal{U}_{g,f} = \mathcal{U}$. Finally, we discuss the analog of the normalizer conjecture and other related questions.

2.2.1 The Normalizers of \mathcal{U}_f

In this subsection, we introduce the generalized unitary units and we prove that they form a subgroup $\mathcal{U}_{g,f}$ of \mathcal{U} which is exactly the normalizer of \mathcal{U}_f . We also study the second normalizer and the main result is that for a periodic group, the second normalizer is equal to the normalizer.

Let f be an orientation homomorphism(possibly trivial) and \mathcal{C} be the centre of $\mathcal{U}(\mathbb{Z}G)$. If $u \in \mathcal{U}(\mathbb{Z}G)$, satisfies $uu^f \in \mathcal{C}$, we call u a generalized f -unitary unit (or for short, generalized unitary unit). We denote the set of all such units by $\mathcal{U}_{g,f}(\mathbb{Z}G)$ (sometimes just $\mathcal{U}_{g,f}$) and now show that $\mathcal{U}_{g,f}(\mathbb{Z}G)$ is the normalizer of $\mathcal{U}_f(\mathbb{Z}G)$.

Theorem 2.2.1. $\mathcal{U}_{g,f}(\mathbb{Z}G)$ is the normalizer of $\mathcal{U}_f(\mathbb{Z}G)$ in $\mathcal{U}(\mathbb{Z}G)$.

Proof. Let $u \in \mathcal{U}_f, v \in \mathcal{U}_{g,f}$. If $w = v^{-1}uv$, then $w^f = v^f u^f v^{-f}$. Hence $ww^f =$

$v^{-1}uvv^fu^fv^{-f} = v^{-1}vv^fu^fu^fv^{-f} = \pm v^{-1}vv^fv^{-f} = \pm 1$. Therefore $w \in \mathcal{U}_f$ and $\mathcal{U}_f \triangleleft \mathcal{U}_{g,f}$.

Conversely, assume $v \in N_{\mathcal{U}}(\mathcal{U}_f)$, the normalizer of \mathcal{U}_f in \mathcal{U} . For any $u \in \mathcal{U}_f$, $(v^{-1}uv)(v^fu^fv^{-f}) = \pm 1$. Therefore $uvv^fu^f = \pm vv^f$. Let $u = g \in G$. We obtain that $gvv^fg^f = \pm vv^f$ and augmentation arguments tell us $gvv^f = vv^fg \ \forall g \in G$. Hence $vv^f \in \mathcal{C}$; therefore, $v \in \mathcal{U}_{g,f}$. \square

Corollary 2.2.2. (Sehgal and Bovdi [14]). $\mathcal{U}_f(\mathbb{Z}G)$ is a normal subgroup of $\mathcal{U}(\mathbb{Z}G)$ if and only if, $\mathcal{U}_{g,f}(\mathbb{Z}G) = \mathcal{U}(\mathbb{Z}G)$.

Proposition 2.2.3. $\forall v \in N_{\mathcal{U}}(G), vv^f \in \mathcal{C}$; therefore, $N_{\mathcal{U}}(G) \subset \mathcal{U}_{g,f}(\mathbb{Z}G)$.

Proof. $\forall v \in N_{\mathcal{U}}(G), \forall g \in G$, we have $v^{-1}gv \in G$. Therefore, $(v^{-1}gv)(v^{-1}gv)^f = \pm 1$. It follows that $v^{-1}gvv^fg^fv^{-f} = \pm 1$, thus $gvv^f = \pm vv^fg$ and augmentation arguments tell us $gvv^f = vv^fg$. Hence $vv^f \in \mathcal{C}$. \square

Now we are going to study the second normalizer, $N_{\mathcal{U}}(\mathcal{U}_{g,f}(\mathbb{Z}G))$. Our main result is that for any periodic group G , the second normalizer of the subgroup of unitary units is equal to the first one.

Theorem 2.2.4. For any periodic group G , $N_{\mathcal{U}}(\mathcal{U}_{g,f}(\mathbb{Z}G)) = \mathcal{U}_{g,f}(\mathbb{Z}G)$.

First we prove several preliminary results.

Lemma 2.2.5. $\forall x_i \in \mathbb{Z}G$, if $\sum_{i=1}^n \sigma_i x_i x_i^f = \pm g$, where $g \in G, \sigma_i = \pm 1$, then $g = 1$.

Proof. Let $x_i = \sum a_{i,j} g_{i,j}$, $x_i^f = \sum a_{i,j} g_{i,j}^f$. Then

$$\begin{aligned} x_i x_i^f &= \sum \pm a_{i,j}^2 + \sum_{j_1 \neq j_2} a_{i,j_1} a_{i,j_2} g_{i,j_1} g_{i,j_2}^f \\ &= \sum \pm a_{i,j}^2 + \sum_{j_1 < j_2} (a_{i,j_1} a_{i,j_2} g_{i,j_1} g_{i,j_2}^f + a_{i,j_2} a_{i,j_1} g_{i,j_2} g_{i,j_1}^f) \end{aligned}$$

Therefore,

$$\sum (\sigma_i x_i x_i^f) = \sum (\sigma_i \sum \pm a_{i,j}^2) + \sum (\sigma_i \sum_{j_1 < j_2} a_{i,j_1} a_{i,j_2} (g_{i,j_1} g_{i,j_2}^f + g_{i,j_2} g_{i,j_1}^f)) = \pm g$$

Taking the augmentations of both sides, we obtain

$$z_0 + z_1 = \pm 1$$

where $z_0 = \sum (\sigma_i \sum \pm a_{i,j}^2)$, $z_1 = \sum (\sigma_i \sum_{j_1 < j_2} a_{i,j_1} a_{i,j_2} (f(g_{i,j_1}) + f(g_{i,j_2})))$. Note that $f(g_{i,j_1}) + f(g_{i,j_2})$ is either ± 2 or 0 . It follows that z_1 is an even number. Hence $z_0 \neq 0$, and this forces $g = 1$. \square

Corollary 2.2.6. $\forall u \in \mathbb{Z}G$, if $uu^f = \pm g$, then $g = 1$; therefore, u is a unitary unit.

Proposition 2.2.7. For any group G , $T(\mathcal{U}_{g,f}) = T(\mathcal{U}_f)$, where T denotes the subset of all torsion elements.

Proof. We only need to prove that $T(\mathcal{U}_{g,f}) \subset T(\mathcal{U}_f)$. $\forall u \in T(\mathcal{U}_{g,f})$, $uu^f = c \in C$. Since $uu^f = u^f u$, we conclude that $o(c) < \infty$, so $c = \pm g$ ([57], p.46, Corollary 1.7). By Corollary 2.2.6, $uu^f = \pm g$ implies that $g = 1$. This leads to $u \in T(\mathcal{U}_f)$ and finishes the proof. \square

Recall that if f is trivial, then $\mathcal{U}_f(\mathbb{Z}G) = \pm G$; therefore, $N_{\mathcal{U}}(\pm G) = \mathcal{U}_{g,f}(\mathbb{Z}G)$ by Theorem 2.2.1. As a consequence we obtain the following Corollary.

Corollary 2.2.8. $T(N_{\mathcal{U}}(G)) = \pm T(G)$.

Remark 2.2.9. *Generally speaking, $T(\mathcal{U}_f)$ is not necessarily equal to $\pm T(G)$. We will illustrate this in an example later.*

Now we are ready to prove our main result (Theorem 2.2.4).

Proof. Let $v \in N(\mathcal{U}_{g,f})$ and $g \in G \subset T(\mathcal{U}_{g,f})$. Since $v^{-1}gv \in T(\mathcal{U}_{g,f})$, we conclude that $v^{-1}gv \in T(\mathcal{U}_f)$ by Proposition 2.2.7. It follows that $\pm 1 = v^{-1}gv(v^{-1}gv)^f = v^{-1}gvv^fg^fv^{-f}$. Augmentation arguments tell us that $gvv^f = vv^fg$. Hence $vv^f \in \mathcal{C}$ and therefore $v \in \mathcal{U}_{g,f}$. This completes the proof. \square

Corollary 2.2.10. *For any periodic group G , $\mathcal{U}_{g,f}(\mathbb{Z}G)$ is a normal subgroup $\mathcal{U}(\mathbb{Z}G)$ if and only if*
 $\mathcal{U}_{g,f}(\mathbb{Z}G) = \mathcal{U}(\mathbb{Z}G)$.

We close this subsection by indicating some results for $N(\mathcal{U}_{g,f})$ when G is arbitrary.

Proposition 2.2.11. *For an arbitrary group G , we have $v \in N(\mathcal{U}_{g,f})$ if and only if $\forall u \in \mathcal{U}_{g,f}, \exists c \in \mathcal{C}$, such that $u(vv^f) = c(vv^f)u$, and $c = c^f$.*

Proof.

$$\begin{aligned}
v \in N(\mathcal{U}_{g,f}) &\iff \forall u \in \mathcal{U}_{g,f}, v^{-1}uv \in \mathcal{U}_{g,f} \\
&\iff v^{-1}uv(v^{-1}uv)^f = v^{-1}u(vv^f)u^fv^{-f} \in \mathcal{C} \\
&\iff uvv^f = c_1vv^fu^{-f}, \text{ for some } c_1 \in \mathcal{C} \\
&\iff uvv^f = cvv^fu, \text{ for some } c \in \mathcal{C} (*) \\
&\iff [vv^f, \mathcal{U}_{g,f}] \subset \mathcal{C}.
\end{aligned}$$

Up to (*), we have proved the first part. Next we will prove that $c = c^f$.

Taking f of both sides of (*), we arrive at

$$vv^f u^f = u^f vv^f c^f$$

Multiplying by (*), we obtain

$$u(vv^f)^2 u^f = cc^f (vv^f)^2 uu^f.$$

On the other hand,

$$u(vv^f)^2 u^f = cvv^f uvv^f u^f = cvv^f cvv^f uu^f = c^2 (vv^f)^2 uu^f \text{ (by (*))}$$

Therefore, we obtain $cc^f = c^2$; i.e. $c = c^f$. This finishes the proof. \square

Corollary 2.2.12. *For an arbitrary group G , if $v \in N(\mathcal{U}_{g,f})$, then either $o(vv^f) = \infty$, or $(vv^f)^2 = 1$.*

Proof. Let $v \in N(\mathcal{U}_{g,f})$. If $o(vv^f) < \infty$, then $o(vv^f)^2 < \infty$. We first prove that $(vv^f)^2 \in \mathcal{C}$.

Recall from Proposition 2.2.11 that $uvv^f = cvv^f u$ if $u \in \mathcal{U}_{g,f}$, and $c = c^f$. Suppose $(vv^f)^n = 1$. We obtain

$$u = u(vv^f)^n = cvv^f u(vv^f)^{n-1} = \dots = c^n (vv^f)^n u = c^n u$$

It follows that $c^n = 1$ and $c = \pm g$. Since $c = c^f$, $c^2 = cc^f = gg^{-1} = 1$. It turns out that

$$u(vv^f)^2 = c^2 (vv^f)^2 u = (vv^f)^2 u, \forall u \in \mathcal{U}_{g,f}$$

Therefore $(vv^f)^2 \in \mathcal{C}$. Now we will prove that $(vv^f)^2 = 1$.

Since $o((vv^f)^2) < \infty$ and $(vv^f)^2 \in \mathcal{C}$, $(vv^f)^2 = g_0$. By Lemma 2.2.5, $g_0 = 1$, and this completes the proof. \square

Remark 2.2.13. *If $v \in N(\mathcal{U}_{g,f})$, then either $o(vv^f) = \infty$, or $vv^f \in \mathcal{U}_f$ and $o(vv^f) \leq 2$.*

2.2.2 Conditions for $\mathcal{U} = \mathcal{U}_{g,f}$

In this subsection, we discuss some necessary and sufficient conditions for $\mathcal{U} = \mathcal{U}_{g,f}$. We also characterize when \mathcal{B}_2 , the subgroup generated by all the bicyclic units, is f -generalized unitary.

Proposition 2.2.14. *For any periodic group, the following conditions are equivalent:*

- (1) $\mathcal{U} = \mathcal{U}_{g,f}$;
- (2) $\mathcal{U}_{g,f}$ is a normal subgroup of \mathcal{U} ;
- (3) $\forall v \in \mathcal{U}, \forall u \in \mathcal{U}_{g,f}, \exists c \in \mathcal{C} \ni u(vv^f) = c(vv^f)u$ and $c = c^f$;
- (4) \mathcal{U}_f is a normal subgroup of \mathcal{U} .

Proof. (1) \iff (2) (by Corollary 2.2.10), (2) \iff (3) (by Proposition 2.2.11), (1) \iff (4) (by Corollary 2.2.2). \square

We note that when f is trivial, the question of when $\mathcal{U} = \mathcal{U}_{g,f}$ reduces to when G is normal in $\mathcal{U}(\mathbb{Z}G)$ which is settled by Cliff and Sehgal [17]. We are interested in the question with nontrivial f and let us first state a result proved by Bovdi and Sehgal [14] describing when $\mathcal{U}(\mathbb{Z}G) = \mathcal{U}_{g,f}(\mathbb{Z}G)$.

Theorem 2.2.15. *Let $f : G \longrightarrow \mathcal{U}(\mathbb{Z})$ be a nontrivial homomorphism with kernel A and let $\mathcal{U}(\mathbb{Z}G)$ be generalized f -unitary. Then G contains an element b such that $G = \langle A, b \rangle$ and one of the following conditions is fulfilled:*

(1) A is an abelian group, the order of the element b divides 4 and $bab^{-1} = a^{-1}$ for all $a \in A$;

(2) A is a Hamiltonian 2-group and G is the semidirect product of A and $\langle b \mid b^2 = 1 \rangle$, and every subgroup of A is normal in G ;

(3) A is Hamiltonian 2-group and G is a semidirect product of a Hamiltonian 2-group and the cyclic group $\langle b \rangle$ of order 4;

(4) $t(A)$ is a central subgroup of G , $t(A)$ is the direct product of $\langle b^2 \mid b^8 = 1 \rangle$ and a group of exponent 2 and $bab^{-1} = a^{-1}b^{4i}$ for all $a \in A$, where i depends on a ;

(5) $t(G)$ is a subgroup, every subgroup of $t(G)$ is normal in G and $t(G)$ satisfies one of the following conditions:

(5.1) $t(G)$ is a Hamiltonian 2-group;

(5.2) $t(G)$ is abelian, the centralizer of $C_G(t(A))$ of $t(A)$ is a subgroup of index 2 in G , and $gag^{-1} = a^{-1}$ for all $a \in t(A)$ and $g \in G \setminus C_G(t(A))$;

(5.3) $t(G)$ is abelian and $t(A)$ is a subgroup of the center of G .

Conversely, let G satisfy one of the conditions (1) -(3) or (5). Further suppose that in case (1) $b^2 = 1$ and in case (5) $G/t(G)$ is a right ordered group. If $f : G \rightarrow \mathcal{U}(\mathbb{Z})$ is a homomorphism with kernel A , then $\mathcal{U}(\mathbb{Z}G)$ is generalized f -unitary.

We will extend Theorem 2.2.15 by proving the sufficiency of case (1) in general.

Proposition 2.2.16. *Let $f : G \rightarrow \mathcal{U}(\mathbb{Z})$ be a nontrivial homomorphism with kernel A . Then G contains an element b such that $G = \langle A, b \rangle$. Suppose that A is an abelian group, the order of the element b divides 4 and $bab^{-1} = a^{-1}$ for all $a \in A$. Then $\mathcal{U}(\mathbb{Z}G)$ is generalized f -unitary.*

Proof. First note that $f(b) = -1$ and $b^2 \in A$. Say $u = a_1 + a_2b \in \mathcal{U}(\mathbb{Z}G)$ where

$a_i \in \mathbb{Z}A$, so $u^f = a_1^* - a_2 b^{-1}$. Let us consider

$$v = uu^f = a_1 a_1^* - a_2 a_2^* + a_1 a_2 b(1 - b^{-2})$$

If $o(b) = 2$, then $uu^f \in \mathcal{C}$; therefore, $u \in \mathcal{U}_{g,f}$. Next we suppose that $o(b) = 4$.

$$\begin{aligned} v^* &= (uu^f)^* = a_1 a_1^* - a_2 a_2^* - a_1 a_2 b(1 - b^2) \\ vv^* &= (a_1 a_1^*)^2 + (a_2 a_2^*)^2 - 2(a_1 a_1^* a_2 a_2^*) b^2 \\ &= (a_1 a_1^* - a_2 a_2^* b^2)^2 = c^2 \end{aligned}$$

where $c = (a_1 a_1^* - a_2 a_2^* b^2) = c^* = c^f \in \mathcal{C}$.

Let $v_1 = vc^{-1}$, thus

$$v_1 v_1^* = vc^{-1}(c^{-1})^* v^* = vv^* c^{-1}(c^*)^{-1} = 1$$

We conclude that $v_1 = \pm g$ for some $g \in G$ and $v = \pm cg$.

Let $g = ab^i$, $a \in A, i = 0, 1$. If $i = 1$, then $g = ab$ and $v = \pm cab$; therefore,

$$\pm c = a^{-1} v b^3 = a^{-1} (a_1 a_1^* - a_2 a_2^*) b^3 + a^{-1} (a_1 a_2 (1 - b^2)) \in \mathcal{C}$$

Since $c \in \mathbb{Z}A$ (by the definition above), we have $a^{-1} (a_1 a_1^* - a_2 a_2^*) b^3 = 0$. However, this is a contradiction to $aug(a_1 a_1^* - a_2 a_2^*) = \pm 1$. As a consequence, $i = 0$ and $g = a$.

Now since

$$a^{-1} (a_1 a_1^* - a_2 a_2^*) + a^{-1} (a_1 a_2 (1 - b^2) b) = a^{-1} v = \pm c \in \mathbb{Z}A,$$

we conclude that $a^{-1} (a_1 a_2 (1 - b^2) b) = 0$, so $(a_1 a_2 (1 - b^2) b) = 0$. Hence it follows that $v = uu^f = a_1 a_1^* - a_2 a_2^* \in \mathcal{C}$. Finally, we have proved that $u \in \mathcal{U}_{g,f}$. \square

Remark 2.2.17. *The sufficiency of case (4) of Theorem 2.2.15 still remains open.*

For any periodic group G , we have now obtained the following necessary and sufficient conditions for $\mathcal{U}(\mathbb{Z}G)$ being generalized f -unitary.

Theorem 2.2.18. *Let G be a periodic group and assumptions be the same as those in Theorem 2.2.15. Then $\mathcal{U}(\mathbb{Z}G)$ is generalized f -unitary if and only if one of the following conditions is satisfied:*

- (1) *A is an abelian group, the order of the element b divides 4 and $bab^{-1} = a^{-1}$ for all $a \in A$;*
- (2) *A is a Hamiltonian 2-group and G is the semidirect product of A and $\langle b \mid b^2 = 1 \rangle$, and every subgroup of A is normal in G ;*
- (3) *A is a Hamiltonian 2-group and G is a semidirect product of a Hamiltonian 2-group and the cyclic group $\langle b \rangle$ of order 4;*
- (4) *G is an abelian group.*

We will now investigate when $\mathcal{B}_2(\mathbb{Z}G)$, the subgroup generated by all bicyclic units of $\mathcal{U}(\mathbb{Z}G)$, is nontrivial and f -generalized unitary. Our main result is that this reduces to the unitary case.

Proposition 2.2.19. *If $\mathcal{B}_2 \subset \mathcal{U}_{g,f}$, then $\mathcal{B}_2 \subset \mathcal{U}_f$.*

Proof. Suppose that $\mathcal{B}_2 \subset \mathcal{U}_{g,f}$. First we prove that $\forall a \in A$, where $A = \ker(f)$, with $o(a) = n < \infty$, $\langle a \rangle$ is a normal subgroup of G . Let us consider a bicyclic unit $u_{a,g} = 1 + (1 - a)g\hat{a}$. Then $u_{a,g}^{-1} = 1 - (1 - a)g\hat{a}$, and $u_{a,g}^f = 1 + \hat{a}g^f(1 - a^{-1})$. Now $u_{a,g}u_{a,g}^f = c \in \mathcal{C}$, so $u_{a,g}^f = cu_{a,g}^{-1}$. We have

$$1 + \hat{a}g^f(1 - a^{-1}) = c - c(1 - a)g\hat{a}.$$

Multiplying by \hat{a} from the right we obtain

$$c\hat{a} - nc(1-a)g\hat{a} = \hat{a}. \quad (2.1)$$

Multiplying by \hat{a} from the left we have $nc\hat{a} = n\hat{a}$. Therefore $c\hat{a} = \hat{a}$. Substituting this into (2.1) we obtain $(1-a)g\hat{a} = 0$. As a consequence, $u_{a,g}$ is trivial and this finishes the first part.

Now we consider any d of finite order in $G \setminus A$. We note that the order of d is always even and $d^2 \in t(A)$ because $f(d)^{o(d)} = (-1)^{o(d)} = 1$. Since $\langle d^2 \rangle$ is normal, \hat{d}^2 is central in $\mathbb{Z}G$. Let $u_{d,g} = 1 + (1-d)g\hat{d} = 1 + (1-d)g(1+d)\hat{d}^2$, thus

$$u_{d,g}^f = 1 - d^{-1}(1-d)g^f(1+d)d^{-1}\hat{d}^2$$

Since $u_{d,g} \in \mathcal{U}_{g,f}$, $u_{d,g}u_{d,g}^f = c \in \mathcal{C}$. We have

$$c - c(1-d)g(1+d)\hat{d}^2 = 1 - d^{-1}(1-d)g^f(1+d)d^{-1}\hat{d}^2.$$

Multiplying by $1+d$ from the left we obtain $c(1+d) = 1+d$. Multiplying by $1-d$ from the right we obtain $c(1-d) = 1-d$. Combining these two equations we have $c = 1$. Therefore $u_{d,g} \in \mathcal{U}_f$. We are done. \square

Combining Proposition 2.2.19 with Theorem 2 in Bovdi and Sehgal [15], we obtain the following theorem :

Theorem 2.2.20. *Let $f : G \longrightarrow \mathcal{U}(\mathbb{Z})$ be a nontrivial orientation homomorphism with kernel A , then $G = \langle A, b \rangle$ where $b \in G$. The subgroup $B_2(\mathbb{Z}G)$ is nontrivial and generalized f -unitary if and only if, G is a non Hamiltonian group which contains an element $b \neq 1$ such that one of the following conditions is satisfied:*

- (1) A is an abelian group, the order of the element b divides 4 and $bab^{-1} = a^{-1}$ for all $a \in A$;
- (2) A is a Hamiltonian 2-group and G is the semidirect product of A and $\langle b \mid b^2 = 1 \rangle$, and every subgroup of A is normal in G ;
- (3) A is a Hamiltonian 2-group and G is the direct product of a Hamiltonian 2-subgroup of A and cyclic group $\langle b \rangle$ of order 4;
- (4) $t(A)$ is an abelian group, every subgroup of $t(A)$ is normal in G and $bab^{-1} = a^{-1}b^{4i}$ for all $a \in A$, where the integer i depends on a .

Finally we make the following observation (compare with Proposition 2.1.4).

Proposition 2.2.21. *For any bicyclic unit u , if u is a unitary unit, then it must be one of the first class unitary units, i.e. $uu^f = 1$.*

Proof. Let $u = 1 + (1 - a)b\hat{a} \in \mathcal{U}_f$. Thus $uu^f = \pm 1$. If $a \in \text{Ker}(f)$, then $\text{aug}(u^f) = 1 + \text{aug}[\hat{a}^f b^f (1 - a^{-1})] = 1$. Therefore, $\text{aug}(uu^f) = 1$, thus $uu^f = 1$.

Now suppose that $a \notin \text{Ker}(f)$. It follows from the proof of Proposition 2.2.19 that $o(a) = 2l$. Note that $\hat{a}^f = 1 - a^{-1} + (a^2)^{-1} - (a^3)^{-1} + \dots + (a^{2l-2})^{-1} - (a^{2l-1})^{-1}$. We have $\text{aug}(\hat{a}^f) = 0$. Therefore, $\text{aug}(u^f) = 1 + \text{aug}[\hat{a}^f b^f (1 + a^{-1})] = 1$. As a consequence $\text{aug}(uu^f) = 1$ and $uu^f = 1$. \square

2.2.3 Conditions for $\mathcal{U}_{g,f} = \mathcal{U}_f$

Theorem 2.2.22. *The following conditions are equivalent;*

- (1) $\mathcal{U}_{g,f} = \mathcal{U}_f$;
- (2) $[\mathcal{U}_{g,f} : \mathcal{U}_f] < \infty$;
- (3) $\forall u \in \mathcal{U}_{g,f}, \exists n \ni u^n \in \mathcal{U}_f$, where n depends on u ;

$$(4) \forall u \in \mathcal{U}_{g,f}, \exists n \ni (uu^f)^n \in \mathcal{U}_f;$$

$$(5) \forall c \in \mathcal{C}, cc^f = \pm 1.$$

The proof is similar to that of Theorem 2.1.6

As we mentioned before, if $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ is trivial, then $\mathcal{U}_{g,f} = \mathcal{U}_f$ (by Lemma 2.2.5). For finite groups, necessary and sufficient conditions for $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ to be trivial were obtained by Ritter and Sehgal (see Theorem 3.1.1 in the next chapter). However, the following Example tells us that $\mathcal{U}_{g,f} = \mathcal{U}_f$ is not sufficient to guarantee that the center of the unit group is trivial.

Example 2.2.23. Let $G = C_6 \times C_4$ where $C_6 = \langle c_1 \rangle$, $C_4 = \langle c_2 \rangle$, and $A = \langle c_1 \rangle \times \langle c_2^2 \rangle$. Then $\mathcal{U}(\mathbb{Z}G) = \mathcal{U}_f$ by (5.2) of Theorem 2.1.8, but there exists a nontrivial central unit (a Bass cyclic unit constructed by a group element of order 12).

However, we have the following sufficient conditions for $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ being trivial.

Proposition 2.2.24. Let $G = \langle A, b \rangle$, where $A = \ker f$ and $f(b) = -1$. If $\mathcal{U}_f = \mathcal{U}_{g,f}$ and one of following conditions holds:

- (1) $b^2 = 1$ and A is abelian ;
- (2) $b^2 = 1$ and for all $a \in A, ab = ba$;
- (3) for all $a \in A, bab^{-1} = a^{-1}$ (so A is abelian);

then $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ is trivial.

Proof. (1) Let $u = a_1 + a_2b \in \mathcal{C}$ where $a_1, a_2 \in \mathbb{Z}A$. Since $ub = bu$, we conclude that $a_i b = ba_i$ for $i = 1, 2$. Hence it also follows that $a_i^* b = ba_i^*$. Now $u^f = a_1^* - ba_2^* = a_1^* - a_2^* b$. Hence $uu^f = a_1 a_1^* - a_2 a_2^* + (a_2 a_1^* - a_1 a_2^*)b$. Since $u \in \mathcal{C} \subset \mathcal{U}_{g,f} = \mathcal{U}_f$, we have $uu^f = \pm 1$. Thus $a_1 a_1^* - a_2 a_2^* = \pm 1$ and $a_2 b a_1^* - a_1 b a_2^* = 0$.

Let $v = a_1 + a_2, v_1 = a_1^* - a_2^*$. Then the above tells us that $vv_1 = a_1a_1^* - a_2a_2^* + (a_2a_1^* - a_1a_2^*) = \pm 1$. Since v commutes with b , we have that v is a central unit in $\mathbb{Z}G$ and hence is contained in $\mathcal{U}_{g,f} = \mathcal{U}_f$, so $vv^f = \pm 1$. But also $v \in \mathbb{Z}A$, so $v^f = v^*$. We conclude that $vv^* = vv^f = 1$. This means that v is trivial, and also that $v^* = \pm v_1$. We conclude that either $a_1 = 0, a_2 = \pm g$ or $a_1 = \pm g, a_2 = 0$ for some $g \in G$. The result follows.

(2) Copying the proof of the first part of (1), we obtain that $vv_1 = \pm 1$. We only need to prove that $v = a_1 + a_2$ is a central unit. By the assumption, $ba_i = a_i b$ so that we only need to verify for all a in A , $aa_i = a_i a$. Notice that $au = ua$ where $u = a_1 + a_2 b$ is a central unit, and this gives that $aa_1 + aa_2 b = a_1 a + a_2 ab$, thus $aa_i = a_i a$. Copying the rest of proof of (1), we finish the proof.

(3) By the proposition in Bovdi and Sehgal [15], for all u in \mathcal{C} , $u \in \mathcal{U}(\mathbb{Z}A)$, thus $u^* = u^f$. Since $\mathcal{C} \subset \mathcal{U}_{g,f} = \mathcal{U}_f$, $uu^* = uu^f = 1$. This implies that u is a trivial unit. ■

Corollary 2.2.25. *Let $G = \langle A, b \rangle$ be a finite group, where $A = \ker f$ and $f(b) = -1$. If $\mathcal{U}_f = \mathcal{U}_{g,f}$ and one of following conditions holds:*

- (1) $b^2 = 1$ and A is abelian ;
- (2) $b^2 = 1$ and $\forall a \in A, ab = ba$;
- (3) $\forall a \in A, bab^{-1} = a^{-1}$ (so A is abelian);

then $\forall g \in G, n$ relatively prime to the order of G, g^n is conjugate to either g or g^{-1} .

Proof. The result follows from Proposition 2.2.24 and Theorem 3.1.1. □

2.2.4 Other Results

We begin this subsection by studying an analog of the normalizer conjecture. Then we continue the investigation initiated in Subsection 2.1 and establish relationships between generalized unitary units in $\mathbb{Z}G$ and $\mathbb{Z}(G \times C_2)$.

A well-known open problem in group rings is the normalizer conjecture:

Conjecture 2.2.26. (*Sehgal [58], Problem 43*) *Let G be finite. Then $N_{\mathcal{U}}(G) = \mathcal{C}G$.*

This conjecture was first proved by Coleman [18] for nilpotent groups and then proved by Jackowski and Marciniak for groups of odd order. In fact, Jackowski and Marciniak [25] have proved this result for groups having a normal Sylow 2-group, simultaneously extending the above results. In general, the problem remains open.

Note that $G \subset \mathcal{U}_f$ and if f is trivial, then $\mathcal{U}_f = \pm G$. Also recall that the normalizer of the subgroup of unitary units in $\mathcal{U}(\mathbb{Z}G)$ is just the subgroup of the generalized unitary units by Theorem 2.2.1. So a natural analog of the normalizer conjecture is that if G is a finite group, then $\mathcal{U}_{g,f}(\mathbb{Z}G) = \mathcal{C}\mathcal{U}_f$.

For our needs, we first let $H = \mathcal{U}_{g,f}$, and $H_1 = \mathcal{C}\mathcal{U}_f$, the subgroup generated by all central and unitary units.

Recalling Theorem 2.2.1 we derive the following corollary:

Corollary 2.2.27. *H_1 is a normal subgroup of H .*

Proposition 2.2.28. *H/H_1 is a group of exponent dividing 2.*

Proof. Let $u \in H$. Then if $uu^f = c$, $c^f = (uu^f)^f = (u^f)^f u^f = uu^f = c$. Considering u^2 , we have $u^2(u^2)^f = uuu^f u^f = (uu^f)(uu^f) = c^2$. Let $u_1 = u^2 c^{-1}$, then $u_1 u_1^f =$

$u^2c^{-1}(c^{-1})^f(u^2)^f = u^2(u^2)^fc^{-2} = c^2c^{-2} = 1$. Therefore $u_1 \in \mathcal{U}_f$, thus $u^2 \in H_1$. As a consequence, H/H_1 is a 2-group or a trivial group. \square

Corollary 2.2.29. *Let G be a finite group. H is a subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$ if and only if H_1 is a subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$.*

Proof. By Krempa ([37] Theorem 2.9), $\mathcal{U}(\mathbb{Z}G)$ is finitely generated. We assume that H is a subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$. So H is finitely generated (by the Schreier Method). This means that H/H_1 is finite, so H_1 is a subgroup of finite index in H and in $\mathcal{U}(\mathbb{Z}G)$ as well. \square

The analog of the normalizer conjecture states that for any finite group G , $H = H_1$. We give the following necessary and sufficient conditions for $H = H_1$.

Proposition 2.2.30. *For any integral group ring, $H = H_1$ if and only if, for all $v \in H$, there exists $c_1 \in \mathcal{C}$ such that $vv^f = \pm c_1c_1^f$.*

Proof. Suppose that $H = H_1$. $\forall v \in H, v = uc$ where $u \in \mathcal{U}_f(\mathbb{Z}G), c \in \mathcal{C}$. Thus $vv^f = uc(uc)^f = ucc^fu^f = (cc^f)(uu^f) = \pm cc^f$.

Conversely, assume that $\forall v \in H, vv^f = \pm cc^f$. Let $u = vc^{-1}$, then $uu^f = vc^{-1}c^{-f}v^f = c^{-1}c^{-f}vv^f = \pm c^{-1}c^{-f}cc^f = \pm 1$. Therefore $v = uc \in H_1$. \square

Unfortunately, the analog of the normalizer conjecture fails in general. The following example will show that $H \neq H_1$ for the integral group ring $\mathbb{Z}D_{16}$.

Example 2.2.31. *Let $G = D_{16}$ and $f(a) = 1, f(b) = -1$. Then $\mathcal{U}(\mathbb{Z}D_{16}) = \mathcal{U}_{g,f}(\mathbb{Z}D_{16}) = H$ (by Theorem 2.2.15), but $H \neq H_1$.*

Proof. Take $u = 1 - (1 + a^3 + b - ab)(1 - a^4)$. Thus

$$\begin{aligned} u^f &= 1 + (-1 + a + b - ab)(1 - a^4) \\ uu^f &= 1 + (-2 + a - a^3)(1 - a^4) \\ &= a^4(1 + (1 - a + a^3)(1 - a^4)) = a^4u_0 \end{aligned}$$

Next we will prove that $u \notin H_1$ since $uu^f \neq \pm cc^f$, where c is a central unit.

According to the proposition in Bovdi and Sehgal [15], $\mathcal{C}(\mathcal{U}(\mathbb{Z}D_{16})) \subset \mathcal{U}(\mathbb{Z}A)$ where $A = \langle a \rangle \cong C_8$. By Karpilovsky ([34], p.154, Example 2), $\mathcal{U}(\mathbb{Z}A) = \pm \langle a \rangle \times \langle a^5u_0^{-1} \rangle = \pm \langle a \rangle \times \langle u_0 \rangle$. Therefore $\mathcal{C}(\mathbb{Z}D_{16}) = \pm \langle a^4 \rangle \times \langle u_0 \rangle$ with u_0 nonperiodic. Assume there exists $c_1 = \pm a^{4i}u_0^k$ such that $a^4u_0 = \pm c_1c_1^f$. But notice that $c_1c_1^f = c_1(c_1)^* = c_1^2 = u_0^{2k} \neq \pm a^4u_0$. This contradiction leads to the result. \square

Remark 2.2.32. From Example 2.2.31, we know that there exists $u \in \mathcal{U}(\mathbb{Z}D_{16}) = \mathcal{U}_{g,f}(\mathbb{Z}D_{16})$, with $u \notin \mathcal{U}_f(\mathbb{Z}D_{16})\mathcal{C}(\mathbb{Z}D_{16})$. Thus $u \notin N_{\mathcal{U}}(D_{16})$ which in fact is $D_{16}\mathcal{C}(\mathbb{Z}D_{16})$ since D_{16} is a 2-group (Coleman [18]). Therefore, there exists a group element g such that $u^{-1}gu \notin \pm D_{16}$. Since $o(u^{-1}gu) < \infty$, we have that $u^{-1}gu \in T(\mathcal{U}(\mathbb{Z}D_{16})) = T(\mathcal{U}_{g,f}(\mathbb{Z}D_{16})) = T(\mathcal{U}_f(\mathbb{Z}D_{16}))$ by Proposition 2.2.7. So we conclude that $T(\mathcal{U}_f(\mathbb{Z}D_{16})) \neq \pm D_{16}$.

Note that although $H_1(\mathbb{Z}D_{16})$ is a normal subgroup of finite index of $H(\mathbb{Z}D_{16}) = \mathcal{U}(\mathbb{Z}D_{16})$, $\mathcal{U}_f(\mathbb{Z}D_{16})$ is still a normal subgroup of infinite index of $\mathcal{U}(\mathbb{Z}D_{16})$.

Now we are going to establish relationships between the generalized unitary units in $\mathbb{Z}G$ and $\mathbb{Z}(G \times C_2)$. Recall that a similar study was carried out earlier for unitary units. In particular, the following is an immediate consequence of Proposition 2.1.2.

Proposition 2.2.33. $1 + 2\alpha$ is a central unit in $\mathbb{Z}G$ if and only if $1 + \alpha(1 - c)$ is a central unit in $\mathbb{Z}(G \times C_2)$.

Copying the proof of Proposition 2.1.5, we obtain:

Proposition 2.2.34. $1 + 2\alpha$ is a generalized unitary unit in $\mathcal{U}(\mathbb{Z}G)$ if and only if $1 + \alpha(1 - c)$ is a generalized unitary unit in $\mathcal{U}(\mathbb{Z}(G \times C_2))$.

Lemma 2.2.35. Suppose that a group G is a semi-direct product of two subgroups K and D , where K is normal, denoted by $G = K \rtimes D$. If K_1, D_1 are subgroups of finite index in K, D respectively, then $\langle K_1, D_1 \rangle$, the subgroup generated by K_1 and D_1 , is a subgroup of finite index in G .

Proof. Suppose $K = \bigcup_{i=1}^n k_i K_1$ and $D = \bigcup_{j=1}^m d_j D_1$. $\forall g \in G, g = kd$, where $k \in K, d \in D$. Now $g = kd = kd_j d'$ for some j and $d' \in D_1$. Since K is normal, $kd_j = d_j w$ for some $w \in K$, and $w = k_i k'$ for some i and $k' \in K_1$. Therefore, we obtain $g = kd = kd_j d' = d_j w d' = d_j k_i k' d'$ for some i, j . As a consequence, $G = \bigcup_{i=1, j=1}^{n, m} d_j k_i \langle K_1, D_1 \rangle$. \square

Our main result is as follows:

Theorem 2.2.36. If $\mathcal{U}(\mathbb{Z}G)$ has a generalized unitary subgroup of finite index, then $\mathcal{U}(\mathbb{Z}(G \times C_2))$ has a generalized unitary subgroup of finite index.

Proof. As we pointed out before, $\mathcal{U}(\mathbb{Z}(G \times C_2))$ is a semi-direct product of K by D where $K = \{1 + \alpha(1 - c) | 1 + \alpha(1 - c) \in \mathcal{U}(\mathbb{Z}(G \times C_2))\} \cong \{1 + 2\alpha | 1 + 2\alpha \in \mathcal{U}(\mathbb{Z}G)\} = T$ and $D \cong \mathcal{U}(\mathbb{Z}G)$. We need to show only that $[K : K \cap \mathcal{U}_{g, f_1}(\mathbb{Z}(G \times C_2))] < \infty$ and $[D : D \cap \mathcal{U}_{g, f_1}(\mathbb{Z}(G \times C_2))] < \infty$ by Lemma 2.2.35.

First, we note that since $[\mathcal{U}(\mathbb{Z}G) : \mathcal{U}_{g,f}(\mathbb{Z}G)] < \infty$, $[T : T \cap \mathcal{U}_{g,f}(\mathbb{Z}G)] < \infty$. By Proposition 2.2.34, $K \cap \mathcal{U}_{g,f_1}(\mathbb{Z}(G \times C_2)) \cong T \cap \mathcal{U}_{g,f}(\mathbb{Z}G)$ by the above isomorphism. Therefore $[K : K \cap \mathcal{U}_{g,f_1}(\mathbb{Z}(G \times C_2))] = [T : T \cap \mathcal{U}_{g,f}(\mathbb{Z}G)] < \infty$. We also notice that $D \cap \mathcal{U}_{g,f_1}(\mathbb{Z}(G \times C_2))$ is just the preimage of $\mathcal{U}_{g,f}(\mathbb{Z}G)$ in D . Thus $[D : D \cap \mathcal{U}_{g,f_1}(\mathbb{Z}(G \times C_2))] = [\mathcal{U}(\mathbb{Z}G) : \mathcal{U}_{g,f}(\mathbb{Z}G)] < \infty$ by the assumption. We are done. \square

Similarly, we obtain:

Corollary 2.2.37. *If $\mathcal{U}(\mathbb{Z}G) = \mathcal{U}_{g,f}(\mathbb{Z}G)$, then $\mathcal{U}(\mathbb{Z}(G \times C_2)) = \mathcal{U}_{g,f_1}(\mathbb{Z}(G \times C_2))$*

Corollary 2.2.38. *For any finite group G , if $[\mathcal{U}(\mathbb{Z}G) : \mathcal{U}_f(\mathbb{Z}G)\mathcal{C}(\mathbb{Z}G)] < \infty$ then the same result holds for $\mathbb{Z}(G \times C_2)$.*

Proof. Note that $\mathcal{U}_{g,f_1}(\mathbb{Z}(G \times C_2))$ is a subgroup of finite index in $\mathcal{U}(\mathbb{Z}(G \times C_2))$ by Theorem 2.2.36 and the fact that $H_1 = \mathcal{U}_f(\mathbb{Z}G)\mathcal{C}(\mathbb{Z}G) \subset \mathcal{U}_{g,f}(\mathbb{Z}G)$. The proof is finished by recalling Corollary 2.2.29. \square

Chapter 3

Central Units

3.1 Introduction

There are very few cases known of nonabelian groups G where the group of central units of $\mathbb{Z}G$, denoted $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$, is nontrivial and where the structure of $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$, including a complete set of generators, has been determined. In this chapter, we show that the central units of augmentation 1 in the integral group ring $\mathbb{Z}A_5$ form an infinite cyclic group $\langle u \rangle$, and we explicitly find the generator u . This result has appeared in [39].

First, we recall the following theorem of Ritter and Sehgal [53] giving necessary and sufficient conditions for $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ to be trivial when G is finite, and give a new proof of one direction of this theorem.

Theorem 3.1.1. *Let G be a finite group. All central units of $\mathbb{Z}G$ are trivial if and only if for every $x \in G$ and every natural number j relatively prime to $|G|$, x^j is conjugate to x or x^{-1} .*

We will use the following equivalent version.

Theorem 3.1.2. *Let G be a finite group. All central units of $\mathbb{Z}G$ are trivial if and only if for every $x \in G$ and every natural number j , relatively prime to $|x|$, x^j is conjugate to x or x^{-1} .*

To see these are equivalent, first suppose that the condition in Theorem 3.1.2 holds and let $x \in G$. If $(j, |G|) = 1$, then $(j, |x|) = 1$. Therefore, x^j is conjugate to x or x^{-1} , and the condition in Theorem 3.1.1 also holds.

Conversely, suppose that the condition in Theorem 3.1.1 holds and let $x \in G$ with j relatively prime to $|x|$. Let $|G| = |x|k$ and let $m = \prod p_i$ be the product of all primes p_i such that $p_i | k$, but $p_i \nmid j$ (set $m = 1$ if no such p_i). Observe that $(j + m|x|, |G|) = (j + m|x|, k|x|) = 1$, and hence we have $x^j = x^{j+m|x|}$ is conjugate to x or x^{-1} . We are done.

Next we will give a proof of Theorem 3.1.2 for one direction: “ \Leftarrow ”. We need the following lemma proved in Bass [7] and a remark proved in Jespers, Parmenter and Sehgal [30].

Lemma 3.1.3. *Let G be any finite group. The images of the Bass cyclic units of $\mathbb{Z}G$ under the natural map $j : \mathcal{U}(\mathbb{Z}G) \rightarrow \mathcal{K}_1(\mathbb{Z}G)$ generate a subgroup of finite index.*

Remark 3.1.4. *Let \mathcal{L} denote the kernel of this map j , and \mathcal{B}_1 the subgroup of $\mathcal{U}(\mathbb{Z}G)$ generated by the Bass cyclic units. It follows that there exists an integer m such that $c^m \in \langle \mathcal{B}_1, \mathcal{L} \rangle$ for all $c \in \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$. Since \mathcal{L} is a normal subgroup of $\mathcal{U}(\mathbb{Z}G)$, we can write $c^m = lb_1b_2 \cdots b_k$ for some $l \in \mathcal{L}$ and Bass cyclic units b_i . We note that $\mathcal{L} \cap \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ is trivial.*

Now we are ready to start our proof.

Proof. Suppose that for every $g \in G$ and every natural number i , relatively prime to $|g|$, g^i is conjugate to g or g^{-1} . Let us consider an arbitrary Bass cyclic unit as follows:

$$b = (1 + g + g^2 + \cdots + g^{i-1})^m + \frac{1 - i^m}{|g|} \hat{g}, \text{ where } m = \phi(n), n = |g|.$$

Case 1: Suppose that $g^i = h^{-1}gh$, which we denote as g^h . Then we have

$$\begin{aligned} b^h &= (1 + g^i + g^{2i} + \cdots + g^{i(i-1)})^m + \frac{1 - i^m}{|g|} \hat{g} \\ bb^h &= (1 + g + g^2 + \cdots + g^{i^2-1})^m + \frac{1 - (i^2)^m}{|g|} \hat{g} \\ b^{h^2} &= (1 + g^{i^2} + g^{2i^2} + \cdots + g^{i^2(i-1)})^m + \frac{1 - i^m}{|g|} \hat{g} \\ bb^hb^{h^2} &= (1 + g + g^2 + \cdots + g^{i^3-1})^m + \frac{1 - (i^3)^m}{|g|} \hat{g} \end{aligned}$$

Similarly,

$$bb^hb^{h^2} \cdots b^{h^{m-1}} = (1 + g + g^2 + \cdots + g^{i^m-1})^m + \frac{1 - (i^m)^m}{|g|} \hat{g}$$

Since $i^{\phi(n)} = 1 \pmod{n}$, we have that $g^{i^m} = g$ and $g^{i^m-1} = 1$. It follows that $1 + g + g^2 + \cdots + g^{i^m-1} = k\hat{g} + 1$, where $k = \frac{i^m-1}{n}$. Now, with $C_i^m = \binom{m}{i}$, we have that

$$\begin{aligned} bb^hb^{h^2} \cdots b^{h^{m-1}} &= (k\hat{g} + 1)^m + \frac{1 - (i^m)^m}{|g|} \hat{g} \\ &= (k^m(\hat{g})^m + C_{m-1}^m k^{m-1}(\hat{g})^{m-1} + \cdots + 1) + \frac{1 - (i^m)^m}{|g|} \hat{g} \\ &= (k^m n^{m-1} \hat{g} + C_{m-1}^m k^{m-1} n^{m-2} \hat{g} + \cdots + 1) + \frac{1 - (i^m)^m}{|g|} \hat{g} \end{aligned}$$

$$\begin{aligned}
&= 1 + \frac{(k^m n^m + C_{m-1}^m k^{m-1} n^{m-1} + \dots + 1)\hat{g} - \hat{g}}{n} + \frac{1 - (i^m)^m}{|g|} \hat{g} \\
&= 1 + \frac{(kn + 1)^m - 1}{n} \hat{g} + \frac{1 - (i^m)^m}{n} \hat{g} \\
&= 1 + \frac{(i^m)^m - 1}{n} \hat{g} + \frac{1 - (i^m)^m}{n} \hat{g} \\
&= 1.
\end{aligned}$$

Case 2: Suppose that $g^{-i} = h^{-1}gh$. Then

$$\begin{aligned}
b &= (1 + g + g^2 + \dots + g^{i-1})^m + \frac{1 - i^m}{|g|} \hat{g} \\
b^h &= (1 + g^{-i} + g^{-2i} + \dots + g^{-i(i-1)})^m + \frac{1 - i^m}{|g|} \hat{g} \\
bb^h &= ((g^{i-1})^m (1 + g^{-1} + g^{-2} + \dots + g^{-(i-1)})^m + \frac{1 - i^m}{|g|} \hat{g}) \times \\
&\quad ((1 + g^{-i} + g^{-2i} + \dots + g^{-i(i-1)})^m + \frac{1 - i^m}{|g|} \hat{g}) \\
&= (g^{i-1})^m (1 + g^{-1} + g^{-2} + \dots + g^{-(i^2-1)})^m + \frac{1 - (i^2)^m}{|g|} \hat{g} \\
b^{h^2} &= (b^h)^h = (1 + g^{i^2} + g^{2i^2} + \dots + g^{i^2(i-1)})^m + \frac{1 - i^m}{|g|} \hat{g} \\
bb^h b^{h^2} &= (g^{i-1})^m (g^{i^2(i-1)})^m (1 + g^{-1} + g^{-2} + \dots + g^{-(i^3-1)})^m + \frac{1 - (i^3)^m}{|g|} \hat{g} \\
&= g^{m(i^2+1)(i-1)} (1 + g^{-1} + g^{-2} + \dots + g^{-(i^3-1)})^m + \frac{1 - (i^3)^m}{|g|} \hat{g}
\end{aligned}$$

Similarly,

$$\begin{aligned}
bb^h b^{h^2} \dots b^{h^{m-1}} &= g^{s_m} (1 + g^{-1} + g^{-2} + \dots + g^{-(i^m-1)})^m + \frac{1 - (i^m)^m}{|g|} \hat{g} \text{ for some } s_m \\
&= g^{s_m} ((k\hat{g} + 1)^m + \frac{1 - (i^m)^m}{|g|} \hat{g}) \text{ since } g^{s_m} \hat{g} = \hat{g}
\end{aligned}$$

As before, the sum in the brackets reduces to 1 and we have

$$bb^hb^{h^2} \dots b^{h^{m-1}} = g^{sm} \in G.$$

Therefore there exists s such that $(bb^hb^{h^2} \dots b^{h^{m-1}})^s = 1$.

Because \mathcal{K}_1 is an abelian group, we have proved that for any central unit $c \in \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$, there exists a large integer r such that

$$\begin{aligned} c^r &= (lb_1b_2 \dots b_n)^{r_1} && \text{(by Remark 3.1.4)} \\ &= l'^{r_1} b_1^{r_1} b_2^{r_1} \dots b_n^{r_1} \\ &= l''(b_1b_1^{h_1} \dots b_1^{h_1^{m_1-1}})^{l_1} \dots (b_nb_n^{h_n} \dots b_n^{h_n^{m_n-1}})^{l_n} \\ &= l'' \in \mathcal{L}. \end{aligned}$$

Note that $l'' \in \mathcal{L} \cap \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ is a trivial central unit by Remark 3.1.4. Therefore c is also trivial and we are done. \square

3.2 Main Results

When G is finite, Ritter and Sehgal [55] constructed a finite set of generators for a subgroup of finite index in $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ (see also [54]), while Jespers, Parmenter and Sehgal [30] found a different set of generators which works for finitely generated nilpotent groups (and some others as well). In the latter case, the generators were constructed from Bass cyclic units in $\mathbb{Z}G$ and the construction depended on the existence of a very well behaved finite normal series in G . In general, however, there is no simple procedure known for constructing examples of central units in $\mathbb{Z}G$ (even when Theorem 3.1.2 guarantees their existence). Also there are very few cases of nonabelian groups where $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ is nontrivial and where a complete set of generators has been obtained for $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$.

In this chapter we make some progress on these questions. Since A_5 , the alternating group on 5 letters, is a simple group, the procedure in [30] cannot be used to construct central units in $\mathbb{Z}A_5$. However, if $\alpha = (12345)$ then α and α^{-1} are conjugate to each other but not to $\alpha^2 = \alpha^7$, so Theorem 3.1.2 says that $\mathcal{C}(\mathcal{U}(\mathbb{Z}A_5))$ is nontrivial. We will show that $\mathcal{C}(\mathcal{U}(\mathbb{Z}A_5)) = \pm\langle u \rangle$ where $\langle u \rangle$ is an infinite cyclic group. More significantly, we will explicitly find the generator u , thus obtaining a complete description of $\mathcal{C}(\mathcal{U}(\mathbb{Z}A_5))$.

Recall that whenever R is a commutative ring with 1, the centre of RG is a free R -module with basis consisting of the finite conjugacy class sums in RG .

A_5 has 5 distinct conjugacy classes, and we will denote the corresponding class sums by C_0, C_1, C_2, C_3, C_4 where $C_0 = 1$, C_1 is the sum of elements conjugate to (12345) , C_2 is the sum of elements conjugate to (13524) , C_3 is the sum of all 3-cycles and C_4 is the sum of all elements which are the product of 2 disjoint transpositions. We will need to use the following identities:

$$\begin{aligned}
C_1^2 &= 12 + 5C_1 + C_2 + 3C_3 \\
C_1C_2 &= C_1 + C_2 + 3C_3 + 4C_4 \\
C_1C_3 &= 5C_1 + 5C_2 + 3C_3 + 4C_4 \\
C_1C_4 &= 5C_2 + 3C_3 + 4C_4 \\
C_2^2 &= 12 + C_1 + 5C_2 + 3C_3 \\
C_2C_3 &= 5C_1 + 5C_2 + 3C_3 + 4C_4
\end{aligned}$$

$$\begin{aligned}
C_2C_4 &= 5C_1 + 3C_3 + 4C_4 \\
C_3^2 &= 20 + 5C_1 + 5C_2 + 7C_3 + 8C_4 \\
C_3C_4 &= 5C_1 + 5C_2 + 6C_3 + 4C_4 \\
C_4^2 &= 15 + 5C_1 + 5C_2 + 3C_3 + 2C_4
\end{aligned}$$

which are known, see for example, (Frobenius [19], pp. 1-37).

Suppose $u \in \mathcal{C}(\mathcal{U}_1(\mathbb{Z}A_5))$. Let $u = \sum a_i C_i$ and $u^{-1} = \sum b_i C_i$ where $a_i, b_i \in \mathbb{Z}, 0 \leq i \leq 4$. Since $uu^{-1} = 1$, the identities just stated can be used to give 5 equations, one for each C_i . The augmentation map tells us that $a_0 + 12a_1 + 12a_2 + 20a_3 + 15a_4 = 1$ and similarly for the b_i . Substituting for a_0 and b_0 , we see that the equation arising from C_0 can be ignored as it is a linear combination of the rest. The other 4 equations are:

$$\begin{aligned}
&(1 - 19a_1 - 11a_2 - 15a_3 - 15a_4)b_1 + (-11a_1 + a_2 + 5a_3 + 5a_4)b_2 \\
&\quad + (-15a_1 + 5a_2 + 5a_3 + 5a_4)b_3 + (-15a_1 + 5a_2 + 5a_3 + 5a_4)b_4 = -a_1 \\
&(a_1 - 11a_2 + 5a_3 + 5a_4)b_1 + (1 - 11a_1 - 19a_2 - 15a_3 - 15a_4)b_2 \\
&\quad + (5a_1 - 15a_2 + 5a_3 + 5a_4)b_3 + (5a_1 - 15a_2 + 5a_3 + 5a_4)b_4 = -a_2 \\
&(3a_1 + 3a_2 - 9a_3 + 3a_4)b_1 + (3a_1 + 3a_2 - 9a_3 + 3a_4)b_2 + \\
&\quad (1 - 9a_1 - 9a_2 - 33a_3 - 9a_4)b_3 + (3a_1 + 3a_2 - 9a_3 + 3a_4)b_4 = -a_3 \\
&(4a_2 + 4a_3 - 8a_4)b_1 + (4a_1 + 4a_3 - 8a_4)b_2 + (4a_1 + 4a_2 + 8a_3 - 16a_4)b_3 \\
&\quad + (1 - 8a_1 - 8a_2 - 16a_3 - 28a_4)b_4 = -a_4
\end{aligned}$$

Adding these together, we obtain

$$(1 - 15(a_1 + a_2 + a_3 + a_4))(b_1 + b_2 + b_3 + b_4) = -(a_1 + a_2 + a_3 + a_4)$$

Since we are dealing with integers, this means that $a_1 + a_2 + a_3 + a_4 = 0$ and $b_1 + b_2 + b_3 + b_4 = 0$. Substituting for a_1 and b_1 , and ignoring the first equation which is then a linear combination of the others, we are reduced to

$$\begin{aligned} (1 + 4a_2 - 8a_3 - 8a_4)b_2 + (-8a_2 - 4a_3 - 4a_4)b_3 \\ + (-8a_2 - 4a_3 - 4a_4)b_4 + a_2 = 0 \end{aligned}$$

$$(1 - 12a_3)b_3 + a_3 = 0$$

$$\begin{aligned} (-8a_2 - 4a_3 - 4a_4)b_2 + (-4a_2 - 12a_4)b_3 \\ + (1 - 4a_2 - 12a_3 - 12a_4)b_4 + a_4 = 0 \end{aligned}$$

It follows from the second equation that $1 - 12a_3$ divides a_3 , forcing $a_3 = 0$ and $b_3 = 0$. We are now reduced to

$$\begin{aligned} (1 + 4a_2 - 8a_4)b_2 + (-8a_2 - 4a_4)b_4 &= -a_2 \\ (-8a_2 - 4a_4)b_2 + (1 - 4a_2 - 12a_4)b_4 &= -a_4 \end{aligned}$$

The determinant of the 2×2 matrix arising here is

$$D = 1 - 20(4a_2^2 + 4a_2a_4 - 4a_4^2 + a_4) = -20(2a_2 + a_4)^2 + (10a_4 - 1)^2.$$

We note that $D \neq 0$ and also

$$\begin{aligned} b_2 &= \frac{(2a_2 + a_4)^2 - (5a_4^2 + a_2)}{D} \\ b_4 &= \frac{10a_4^2 - a_4 - 2(2a_2 + a_4)^2}{D} \end{aligned}$$

Since $2b_2 + b_4 = \frac{-2a_2 - a_4}{D}$ is an integer, we have that $D|(2a_2 + a_4)$. The equation for b_4 then says that $D|a_4(10a_4 - 1)$. We conclude from the equation for D that $\gcd(D, a_4) = 1$, so $D|(10a_4 - 1)$. Setting $2a_2 + a_4 = Du$ and $10a_4 - 1 = Dv$, we obtain $D = D^2(v^2 - 20u^2)$. Since $D \neq 0$ and $D \neq -1$, it follows that $D = 1$.

We then have that $(10a_4 - 1)^2 - 20(2a_2 + a_4)^2 = 1$, and this can be rewritten as $(2a_2 + a_4)^2 = (5a_4 - 1)a_4$. It follows that a_4 is an even number and that both a_4 and $5a_4 - 1$ are \pm (perfect squares). Let $a_4 = \pm 4Y^2$ and $5a_4 - 1 = \pm X^2$. If $a_4 > 0$, we get $X^2 - 20Y^2 = -1$. Since the left hand side is 0 or 1 (mod 4), this equation has no solution.

If $a_4 \leq 0$, we have the Pell's equation $X^2 - 20Y^2 = 1$. Working back through the identities which have been developed, we have proved

Proposition 3.2.1. $\mathcal{C}(\mathcal{U}(\mathbb{Z}A_5)) = \{\pm u | u = (1 + 12Y^2)C_0 + (\pm XY + 2Y^2)C_1 + (\mp XY + 2Y^2)C_2 - 4Y^2C_4 \text{ where } X, Y \text{ run through all solutions of the Pell's equation } X^2 - 20Y^2 = 1\}$.

Note that if X, Y is any solution of the above Pell's equation, then the solution $-X, Y$ gives the same units, so we may assume X and Y are nonnegative. Also, if X, Y is a particular solution of the equation, then the 2 units obtained from this solution are inverse to each other - i.e., if $u = (1 + 12Y^2)C_0 + (XY + 2Y^2)C_1 + (-XY + 2Y^2)C_2 - 4Y^2C_4$, then $u^{-1} = (1 + 12Y^2)C_0 + (-XY + 2Y^2)C_1 + (XY + 2Y^2)C_2 - 4Y^2C_4$.

For example, the solution $X = 9, Y = 2$ gives the inverse pair $v = 49 + 26C_1 - 10C_2 - 16C_4$, $v^{-1} = 49 - 10C_1 + 26C_2 - 16C_4$. In fact, our main theorem shows that this particular v is more than just an isolated example.

Theorem 3.2.2. $C(\mathcal{U}(\mathbb{Z}A_5)) = \pm\langle v \rangle$ where v is as defined above.

A careful discussion of solutions to Pell's equation can be found in [42], but for our purposes the crucial result is

Lemma 3.2.3. (Niven and Zuckerman [42], Theorem 7.26) *Consider the Pell's equation $x^2 - dy^2 = 1$ where d is a positive integer which is not a perfect square. Let X_1, Y_1 be the least positive solution to the equation. Then all positive solutions are given by X_n, Y_n for $n = 1, 2, 3, \dots$, where X_n and Y_n are the integers defined by $X_n + Y_n\sqrt{d} = (X_1 + Y_1\sqrt{d})^n$.*

Proof of Theorem 3.2.2

Proof. By Proposition 3.2.1 and the subsequent remark, we are considering nonnegative solutions to the equation $X^2 - 20Y^2 = 1$. When $Y = 0$ we get $u = 1$, while there is no solution when $X = 0$, so we may assume $X, Y > 0$.

All positive solutions are given by X_n, Y_n as stated in Lemma 3.2.3. For each such n , define

$$u_n = 1 + 12Y_n^2 + (X_nY_n + 2Y_n^2)C_1 + (-X_nY_n + 2Y_n^2)C_2 - 4Y_n^2C_4$$

It is easy to see that $X = 9, Y = 2$ is the least positive solution of $X^2 - 20Y^2 = 1$, so $u_1 = v$.

Using our earlier remarks on inverses, we will be finished if we can show that $u_n = u_1^n$ for all $n \geq 1$. This we will do by induction, the case $n = 1$ being obvious. Assume the result is true when $n = k$ for some $k \geq 1$. Since u_1^{k+1} is a central unit, Proposition 3.2.1 tells us that we only need prove that the identity coefficient of u_1^{k+1} equals $1 + 12Y_{k+1}^2$ and that the coefficient of C_1 in u_1^{k+1} equals $X_{k+1}Y_{k+1} + 2Y_{k+1}^2$.

The identity coefficient of $u_1^{k+1} = u_1 u_k$ is $49(1 + 12Y_k^2) + 12(26)(X_k Y_k + 2Y_k^2) + 12(-10)(-X_k Y_k + 2Y_k^2) + 15(-16)(-4Y_k^2) = 49 + 432X_k Y_k + 1932Y_k^2$. On the other hand, Lemma 3.2.3 says that $1 + 12Y_{k+1}^2 = 1 + 12(9Y_k + 2X_k)^2 = 1 + 12(81Y_k^2 + 36X_k Y_k + 4(20Y_k^2 + 1)) = 49 + 432X_k Y_k + 1932Y_k^2$, as desired.

The coefficient of C_1 in u_1^{k+1} equals $49(X_k Y_k + 2Y_k^2) + 26(1 + 12Y_k^2) + 5(26)(X_k Y_k + 2Y_k^2) + 26(-X_k Y_k + 2Y_k^2) + (-10)(X_k Y_k + 2Y_k^2) + (-10)(-X_k Y_k + 2Y_k^2) + 5(-10)(-4Y_k^2) + 5(-16)(-X_k Y_k + 2Y_k^2) + 5(-16)(-4Y_k^2) = 26 + 233X_k Y_k + 1042Y_k^2$. Lemma 3.2.3 says that $X_{k+1}Y_{k+1} + 2Y_{k+1}^2 = (9X_k + 40Y_k)(2X_k + 9Y_k) + 2(2X_k + 9Y_k)^2 = 26(20Y_k^2 + 1) + 233X_k Y_k + 522Y_k^2 = 26 + 233X_k Y_k + 1042Y_k^2$, and this completes the proof. \square

Chapter 4

Hypercentral Units in the Integral Group Ring of a Periodic Group

4.1 Introduction

Let G be a finite group and $\mathcal{U}_1 = \mathcal{U}_1(\mathbb{Z}G)$ be the group of units of augmentation 1 of its integral group ring $\mathbb{Z}G$. Arora, Hales and Passi studied hypercentral units in [3]. Their main result is that the central height of \mathcal{U}_1 is at most 2, i.e. $Z_2(\mathcal{U}_1) = Z_3(\mathcal{U}_1)$, where

$$\{1\} = Z_0(\mathcal{U}_1) \leq Z_1(\mathcal{U}_1) \leq \cdots \leq Z_n(\mathcal{U}_1) \leq \cdots$$

denotes the upper central series of \mathcal{U}_1 . The finiteness of the exponent of $Z_1(G)$ is the key to their proof (see [3], Proposition 2.3). However, this finiteness no longer holds for the integral group ring of a periodic group. In this chapter , we use a different approach to extend their result to the integral group ring of any periodic group. In section 4.3, we establish the relationship between hypercentral units and generalized unitary units.

4.2 Main Results

Let G be an arbitrary group and let

$$\{1\} = Z_0(\mathcal{U}) \leq Z_1(\mathcal{U}) \leq \cdots \leq Z_n(\mathcal{U}) \leq \cdots$$

be the upper central series of the unit group $\mathcal{U}(\mathbb{Z}G)$. Let $\tilde{Z}(\mathcal{U}) = \bigcup_{n=1}^{\infty} Z_n(\mathcal{U})$. Then $\tilde{Z}(\mathcal{U})$ is a normal subgroup of \mathcal{U} and is called the hypercentre of \mathcal{U} . Let G be a periodic group and let $T = T(\tilde{Z}(\mathcal{U}))$ denote the set of all torsion units in $\tilde{Z}(\mathcal{U})$ having augmentation 1. Since $T = \bigcup_{n=1}^{\infty} T(Z_n(\mathcal{U}))$, and $T'_n = \{\pm u | u \in T(Z_n(\mathcal{U}))\}$ is a characteristic subgroup of $Z_n(\mathcal{U})$ for each n , it follows that T is a periodic normal subgroup of $\mathcal{U}(\mathbb{Z}G)$.

Now the results of Bovdi [8, 9] apply to give the following:

Theorem 4.2.1. *Let G be a periodic group. Then exactly one of the following occurs:*

- (1) G is a Hamiltonian 2-group and $T = G$;
- (2) $T = Z_1(G)$;
- (3) G has an Abelian normal subgroup H of index 2 containing an element a of order 4 such that for each $g \in G \setminus H$, $g^2 = a^2$ and $ghg^{-1} = h^{-1}$ for all $h \in H$, and $T = \langle a \rangle \oplus E = Z_2(\mathcal{U}) \cap Z_2(G)$ where E is an elementary Abelian 2-group.

Recalling Theorem 12.5.4 in Hall [20], we have the following:

Remark 4.2.2. *In case (1) of Theorem 4.2.1, $G = Q \oplus E$ where Q is the quaternion group of order 8 and E is an elementary Abelian 2-group. Furthermore, $\mathcal{U}(\mathbb{Z}G) = \pm G$.*

Now we give a proof of Theorem 4.2.1.

Proof. By Bovdi ([8], Theorem 1), $T \subseteq G$ and if T is non-Abelian, then T is a Hamiltonian 2-group and $T \triangleleft G$. In this case, as a result of Bovdi ([9], Theorem 3), $G = T(G)$ is also a Hamiltonian 2-group. Since for a Hamiltonian 2-group, $\mathcal{U}(\mathbb{Z}G) = \pm G$ and $G = Z_2(G)$, this gives $G = T$. It follows that possibility (1) occurs whenever T is non-Abelian.

Next suppose that T is Abelian and $T \subseteq Z_1(G)$. Since $Z_1(G) \subseteq T(Z_1(\mathcal{U})) \subseteq T$, we have $T = Z_1(G)$. Consequently (2) occurs.

Now suppose that T is Abelian and $T \not\subseteq Z_1(G)$. By Bovdi ([9], Theorem 11), G must be of type (3), but not a Hamiltonian 2-group (since T is assumed to be Abelian). In this case, $\langle a \rangle$ is a normal subgroup of $\mathcal{U}(\mathbb{Z}G)$, (see [9], proof of Theorem 11) and hence for any $u \in \mathcal{U}(\mathbb{Z}G)$, $u^{-1}au = a$ or a^3 (since $o(u^{-1}au) = o(a) = 4$). Therefore, $[u, a] \in \langle a^2 \rangle \subseteq \mathcal{C}(\mathcal{U})$. It follows that $a \in Z_2(\mathcal{U})$ and hence $a \in T$. Since T is Abelian and for any $g \notin H$, $[a, g] \neq 1$, we conclude $g \notin T$. Therefore we have a quaternion subgroup of order 8, $Q_8 = \langle a, g \rangle$, such that $Q_8 \cap T = \langle a \rangle$. As a result of Bovdi ([9], Theorem 10(3)), $T = \langle a \rangle \oplus E$ as in case (3). Because $E \subseteq Z_1(G)$, $T = Z_2(G) \cap Z_2(\mathcal{U})$.

For completeness, we include a proof that $\langle a \rangle$ is a normal subgroup of $\mathcal{U}(\mathbb{Z}G)$, where $G = \langle H, b \rangle$ is as in case (3).

Let $x \in \mathcal{U}(\mathbb{Z}G)$, $x = x_1 + x_2b$ where $x_1, x_2 \in \mathbb{Z}H$. Then $axx^* = a[x_1x_1^* + x_2x_2^* + x_1x_2(1+a^2)b] = xx^*a$ since $b^{-1}xb = x^*$ for any $x \in \mathbb{Z}H$. Further, $(x^{-1}ax)(x^{-1}ax)^* = x^{-1}ax(x^*a^*(x^*)^{-1}) = x^{-1}xx^*aa^*(x^*)^{-1} = 1$. Consequently, $x^{-1}ax = g \in G$ and $o(g) = 4$. If $g \notin H$, then $g = hb$ for some $h \in H$. It follows from $ax = xg$, that

$ax_1 + ax_2b = (x_1 + x_2b)g = x_1hb + x_2h^{-1}a^2$. Therefore $ax_1 = x_2h^{-1}a^2$ and an augmentation argument gives $aug(x_1) = aug(x_2)$. Observing that $\pm 1 = aug(x) = aug(x_1) + aug(x_2) = 2aug(x_1)$, we have a contradiction. Hence $g \in H$ and $g^2 = a^2$. We claim that $g \in \langle a \rangle$ and therefore we are done. Otherwise, suppose $g \notin \langle a \rangle$. Then $e = (1 + a + a^2 + a^3)(1 - g) (\neq 0)$ is an element in the center of the group ring $\mathbb{Z}G$ (since $eh = he$ for all $h \in H$ and $be = \hat{a}(1 - g^{-1})b = \hat{a}(1 - g^2g)b = \hat{a}(1 - a^2g)b = \hat{a}(1 - g)b = eb$). It follows that $-e = ge = x^{-1}axe = x^{-1}aex = x^{-1}ex = e$. Hence $e = 0$, which again leads to a contradiction. \square

Corollary 4.2.3. *Let G be a periodic group. Then $T \leq Z_2(G)$.*

Corollary 4.2.4. *Let G be a periodic group. If u is a nontrivial torsion unit, then $u \notin \tilde{Z}(\mathcal{U}(\mathbb{Z}G))$.*

We first prove the following lemma which is needed for proving the main Theorem 4.2.6.

Lemma 4.2.5. *Let G be any periodic group. Then $Z_2(\mathcal{U}(\mathbb{Z}G)) \subseteq N_{\mathcal{U}(\mathbb{Z}G)}(G)$.*

Proof. Let $v \in Z_2(\mathcal{U}(\mathbb{Z}G))$, and $g \in G$. Then $[v, g] = vgv^{-1}g^{-1} = c \in Z_1(\mathcal{U}(\mathbb{Z}G)) = \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$. It follows that $o(cg) = o(vgv^{-1}) < \infty$, and therefore c is of finite order. In view of Sehgal ([57], p.46), we conclude that c is a trivial unit. Consequently, $vgv^{-1} = cg \in G$ and this leads to the desired result. \square

If H and K are subsets of a group G , then we denote by $[H, K]$ the subgroup of G generated by the commutators $[h, k] = hkh^{-1}k^{-1}$, $h \in H$, $k \in K$. Now we prove the main result of this chapter – the central height of the unit group of an integral group ring of a periodic group is at most 2.

Theorem 4.2.6. *Let G be a periodic group. Then $Z_3(\mathcal{U}(\mathbb{Z}G)) = Z_2(\mathcal{U}(\mathbb{Z}G))$.*

Proof. First we prove that $[Z_2(\mathcal{U}(\mathbb{Z}G)), \mathcal{U}(\mathbb{Z}G)] \subseteq Z_1(G)$. (4.2.1)

Note that $Z_2(\mathcal{U}(\mathbb{Z}G))/Z_1(\mathcal{U}(\mathbb{Z}G))$ is periodic since

$$Z_2^2(\mathcal{U}(\mathbb{Z}G))/Z_1(\mathcal{U}(\mathbb{Z}G)) \subseteq N_{\mathcal{U}(\mathbb{Z}G)}^2(G)/Z_1(\mathcal{U}(\mathbb{Z}G)) \subseteq GZ_1(\mathcal{U}(\mathbb{Z}G))/Z_1(\mathcal{U}(\mathbb{Z}G))$$

by Lemma 4.2.5 and Sehgal ([58], Proposition 9.5). It follows that for any $u_2 \in Z_2(\mathcal{U}(\mathbb{Z}G))$, there exists a positive integer $n(u_2)$ such that $(u_2)^{n(u_2)} \in Z_1(\mathcal{U}(\mathbb{Z}G))$. Now for any $u \in \mathcal{U}(\mathbb{Z}G)$, we have that $[u_2, u] = u_2 u u_2^{-1} u^{-1} = c$ so $u u_2^{-1} u^{-1} = u_2^{-1} c$, where c is a central unit. By taking the $n(u_2)$ th power of both sides of the above identity, we obtain that $u u_2^{-n(u_2)} u^{-1} = u_2^{-n(u_2)} c^{n(u_2)}$. This forces $c^{n(u_2)} = 1$ since $u_2^{n(u_2)}$ is a central unit and therefore, $c \in Z_1(G)$ by Sehgal([57], p.46). Finally, we conclude that $[Z_2(\mathcal{U}(\mathbb{Z}G)), \mathcal{U}(\mathbb{Z}G)] \subseteq Z_1(G)$.

Next we prove that $Z_{n+1}^2(\mathcal{U}(\mathbb{Z}G)) \subseteq Z_n(\mathcal{U}(\mathbb{Z}G))$ for all $n \geq 1$. (4.2.2)

We first prove that $Z_2^2(\mathcal{U}(\mathbb{Z}G)) \subseteq Z_1(\mathcal{U}(\mathbb{Z}G))$ by contradiction. Assume that $Z_2^2(\mathcal{U}(\mathbb{Z}G)) \not\subseteq Z_1(\mathcal{U}(\mathbb{Z}G))$. Since $Z_2^2(\mathcal{U}(\mathbb{Z}G)) \subseteq N_{\mathcal{U}(\mathbb{Z}G)}^2(G) \subseteq GZ_1(\mathcal{U}(\mathbb{Z}G))$ as seen earlier, there exists a group element $g \in Z_2(\mathcal{U}(\mathbb{Z}G)) \setminus Z_1(\mathcal{U}(\mathbb{Z}G))$. Let $u \in \mathcal{U}(\mathbb{Z}G)$. Then $[u, g] = g_0 \in Z_1(G)$. Therefore, there exists a positive integer $n = n(u)$ such that $u^n g u^{-n} = g$. It follows from Theorem 1.2 of Parmenter [46] that the exponent of $Z_1(G)$ is 2. Therefore, for all $u_2 \in Z_2(\mathcal{U}(\mathbb{Z}G))$ and all $g' \in G$, we have $[u_2^2, g'] = [u_2, g']^2 = (g'_0)^2 = 1$. This means that u_2^2 is a central unit, forcing $Z_2^2(\mathcal{U}(\mathbb{Z}G)) \subseteq Z_1(\mathcal{U}(\mathbb{Z}G))$. This contradiction finishes the proof.

The proof continues by induction. We just proved that the result is true for $n = 1$. Assume that the result is also true for $n = k - 1 \geq 1$. Now consider the case where $n = k$. Let $u \in \mathcal{U}(\mathbb{Z}G)$ and $u_{k+1} \in Z_{k+1}(\mathcal{U}(\mathbb{Z}G))$. Then $[u_{k+1}, u] = u_k \in$

$Z_k(\mathcal{U}(\mathbb{Z}G))$. It in turn yields that $[u_{k+1}^2, u] = [u_{k+1}, u_k]u_k^2 \in Z_{k-1}(\mathcal{U}(\mathbb{Z}G))$ by the inductive assumption, and therefore we conclude $u_{k+1}^2 \in Z_k(\mathcal{U}(\mathbb{Z}G))$. We are done.

Moreover, in view of the fact that for any $u_3 \in Z_3(\mathcal{U}(\mathbb{Z}G))$, $u \in \mathcal{U}(\mathbb{Z}G)$, $[u_3, u]^2 = [u_3, [u_3, u]]^{-1}[u_3^2, u] \in Z_1(G)$ by (4.2.1) and (4.2.2), we conclude that

$$[Z_3(\mathcal{U}(\mathbb{Z}G)), \mathcal{U}(\mathbb{Z}G)] \subseteq T. \quad (4.2.3)$$

Now we are ready to prove our main result: $Z_3(\mathcal{U}(\mathbb{Z}G)) = Z_2(\mathcal{U}(\mathbb{Z}G))$.

According to Theorem 4.2.1, we need to deal with the following three cases.

(a) Suppose that G is a Hamiltonian 2-group. Then $\mathcal{U}(\mathbb{Z}G) = \pm G = \pm T = Z_2(\mathcal{U}(\mathbb{Z}G))$ and we are done.

(b) Suppose that T is a central subgroup of $\mathcal{U}(\mathbb{Z}G)$. Then $T = Z_1(G)$. The result follows immediately from (4.2.3).

(c) Suppose that T is abelian but not a central subgroup. Then $G = \langle H, g \rangle$ is a group of the type (3) in Theorem 4.2.1 and therefore, $T = \langle a \rangle \oplus E = Z_2(\mathcal{U}(\mathbb{Z}G)) \cap Z_2(G)$. In this case, we first observe that $Z_1(G) = \{x \in G \mid x^2 = 1\}$ and the exponent of T is 4.

Next we prove the following result:

$$[Z_3(\mathcal{U}(\mathbb{Z}G)), \mathcal{B}_2] = 1. \quad (4.2.4)$$

We first show that $[Z_2(\mathcal{U}(\mathbb{Z}G)), \mathcal{B}_2] = 1$. Let $u_2 \in Z_2(\mathcal{U}(\mathbb{Z}G))$ and $u_{b,a} = 1 + (1-b)a\hat{b}$ be a bicyclic unit. Then $[u_2, u_{b,a}] = c_0 \in Z_1(G)$ by (4.2.1). Therefore, there exists a positive integer n such that $[u_2, u_{b,a}]^n = c_0^n = 1$. It in turns yields that $[u_2, u_{b,a}^n] = 1$ since $[u_2, u_{b,a}^n] = [u_2, u_{b,a}]u_{b,a} [u_2, u_{b,a}^{n-1}]u_{b,a}^{-1} = [u_2, u_{b,a}][u_2, u_{b,a}^{n-1}]$ (since

$[u_2, u_{b,a}^{n-1}] \in Z_1(G)$ by (4.2.1)) $= [u_2, u_{b,a}] [u_2, u_{b,a}]^{n-1}$ (by inductive assumption) $= [u_2, u_{b,a}]^n$. Observing that $u_{b,a}^n = 1 + n(1-b)a\hat{b}$, we obtain that $[u_2, u_{b,a}] = 1$ and this leads to the desired result. Next let $u_3 \in Z_3(\mathcal{U}(\mathbb{Z}G))$ and b be a bicyclic unit. Then $[u_3, b] \in T$ by (4.2.3), and hence $[u_3, b]^n = 1$ for some positive integer n . Note that $[u_3, b^n] = [u_3, b^{n-1}]b^{n-1}[u_3, b]b^{-(n-1)} = [u_3, b^{n-1}][b^{n-1}, [u_3, b]][u_3, b]$ and, $[b^{n-1}, [u_3, b]] = 1$ since $[u_3, b] \in Z_2(\mathcal{U}(\mathbb{Z}G))$. We conclude, by induction, that $[u_3, b^n] = [u_3, b^{n-1}][u_3, b] = [u_3, b]^{n-1}[u_3, b] = [u_3, b]^n = 1$. Therefore, $[u_3, b] = 1$ as seen before and we are done.

Now we claim that

$$[Z_3(\mathcal{U}(\mathbb{Z}G)), G] \subseteq \langle a^2 \rangle. \quad (4.2.5)$$

Let $x \in Z_3(\mathcal{U}(\mathbb{Z}G))$. For any group element $h \in H$ for which $h^{-1}gh \in \langle g \rangle$, we observe that $h^{-1}gh = g^i$ and $o(g) = o(g^i)$, forcing $i = 1$ or 3 . Also noticing that $h^{-1}g = gh$, we obtain that $g^i = gh^2$; therefore, either $h^2 = 1$ or $(ah)^2 = 1$ (since $a^2 = g^2$). It follows that either h or ah is in $Z_1(G)$. Since $\langle a \rangle$ is a normal subgroup of $\mathcal{U}(\mathbb{Z}G)$ (see proof of Theorem 4.2.1), we have (for $k = 1$ or 3)

$$[x, ah] = xa(hx^{-1}h^{-1})a^{-1} = xhx^{-1}h^{-1}a^ka^{-1} = [x, h]a^{2j} \text{ for some } j.$$

Hence $[x, h] \in \langle a^2 \rangle$. On the other hand, suppose $h \in H$ and $h^{-1}gh \notin \langle g \rangle$. Then we have a nontrivial bicyclic unit $u_{g,h} = 1 + (1-g)h(1+g+g^2+g^3)$. It follows from (4.2.4), that

$$(1-g)h(1+g+g^2+g^3) = x^{-1}(1-g)h(1+g+g^2+g^3)x.$$

Note that $x^{-1}gx$ and $x^{-1}hx$ are in G by (4.2.3), but $h^{-1}gh \notin \langle g \rangle$. After expanding the above identity, we observe that all the group elements on the left hand side(respectively, on the right hand side) are different. Therefore, we conclude, by augmentation arguments that

$$h(1 + g + g^2 + g^3) = x^{-1}h(1 + g + g^2 + g^3)x. \quad (4.2.6)$$

Consequently, $x^{-1}hx \in \langle g \rangle$ which in turn shows that $[h^{-1}, x^{-1}] \in \langle g \rangle \cap T = \langle a^2 \rangle$ by (4.2.3) and Theorem 4.2.1(3). Thus $[x^{-1}, h] = h[h^{-1}, x^{-1}]h^{-1} \in \langle a^2 \rangle$. Therefore, by setting $y = x^{-1}$, we obtain that $[y, h] \in \langle a^2 \rangle$ for any $y \in Z_3(\mathcal{U}(\mathbb{Z}G))$.

Note that, since G is not a Hamiltonian group, there must exist a group element $h \in H$ such that $h^{-1}gh \notin \langle g \rangle$. Consequently, (4.2.6) yields that

$$\begin{aligned} (1 + g + g^2 + g^3) &= (h^{-1}x^{-1}hx)x^{-1}(1 + g + g^2 + g^3)x \\ &= a^{2i}x^{-1}(1 + g + g^2 + g^3)x \\ &= x^{-1}a^{2i}(1 + g + g^2 + g^3)x \\ &= x^{-1}(1 + g + g^2 + g^3)x \end{aligned}$$

Thus $g = x^{-1}gx$ or $g = x^{-1}g^3x$ and hence $[x, g] \in \langle g^2 \rangle = \langle a^2 \rangle$. Thus (4.2.5) is proved and hence $[Z_3^2(\mathcal{U}(\mathbb{Z}G)), G] = 1$. It follows that

$$Z_3^2(\mathcal{U}(\mathbb{Z}G)) \subseteq Z_1(\mathcal{U}(\mathbb{Z}G)). \quad (4.2.7)$$

Suppose that there exists $x \in Z_3(\mathcal{U}(\mathbb{Z}G)) \setminus Z_2(\mathcal{U}(\mathbb{Z}G))$. Then, for some $u \in \mathcal{U}(\mathbb{Z}G)$, $[x, u] = t \in T$ is an element of order 4. Mapping t into $\mathbb{Z}(G/G')$, we obtain

that $\bar{t} = [\bar{x}, \bar{u}] = \bar{1}$ since $\mathbb{Z}(G/G')$ is a commutative group ring. Thus $t - 1 \in \Delta(G')\mathbb{Z}G$. This implies $t \in G'$, the derived group of G . It is not hard to check that, in this case, $G' = \{h^2 | h \in H\}$. Note that $[x, h] \in \langle a^2 \rangle \subseteq C(\mathcal{U})$ for all $h \in H$ by (4.2.5), so $[x, h]^2 = 1$. It follows that $[x, t] = [x, h^2]$ (for some $h \in H$) = $[x, h]h[x, h]h^{-1} = [x, h]^2 = 1$. Hence $[x^2, u] = x[x, u]x^{-1}[x, u] = [x, t]t^2 = t^2 \neq 1$. However, in view of (4.2.7), we have $[x^2, u] = 1$, a contradiction. Thus we must have $Z_2(\mathcal{U}(\mathbb{Z}G)) = Z_3(\mathcal{U}(\mathbb{Z}G))$ always. \square

We note that the next result follows immediately from (4.2.1).

Corollary 4.2.7. *Let G be a periodic group. If $Z_1(G) = 1$, then $Z_1(\mathcal{U}(\mathbb{Z}G)) = Z_2(\mathcal{U}(\mathbb{Z}G)) = \tilde{Z}(\mathcal{U}(\mathbb{Z}G))$.*

Corollary 4.2.8. *Let G be a periodic group. If all central units are trivial, then all hypercentral units are trivial too.*

Proof. Let $u \in \tilde{Z}(\mathcal{U}(\mathbb{Z}G))$. Then $u \in N_{\mathcal{U}}(G)$ by Lemma 4.2.5 and Theorem 4.2.6. It follows from Sehgal ([58], Proposition 9.4) that $uu^* = g \in Z_1(G)$. Now Lemma 2.2.5 says $uu^* = 1$ and therefore, u is trivial. We are done. \square

By recalling Theorem 3.1.1 of Ritter and Sehgal giving necessary and sufficient conditions for all central units to be trivial when G is finite, we obtain the following necessary and sufficient conditions for all hypercentral units to be trivial.

Corollary 4.2.9. *Let G be a finite group. All hypercentral units of $\mathbb{Z}G$ are trivial if and only if for every $x \in G$ and every natural number j relatively prime to $o(g)$, x^j is conjugate to x or x^{-1} .*

4.3 The Relationship Between Hypercentral Units and Generalized Unitary Units

Recall that in Chapter 2, generalized unitary units were defined in terms of central units. In this section, we first introduce, in terms of hypercentral units of $\mathcal{U}(\mathbb{Z}G)$, an equivalent definition of generalized unitary units of an integral group ring $\mathbb{Z}G$ when G is a periodic group. Then we discuss the relationship between hypercentral units and generalized unitary units. Moreover, we obtain necessary and sufficient conditions for $\mathcal{U}_{g,f} = \tilde{Z}(\mathcal{U})$.

Let G be an arbitrary group, f be an orientation homomorphism, and

$$H = \{u \in \mathcal{U}(\mathbb{Z}G) \mid uu^f \in Z_2(\mathcal{U}(\mathbb{Z}G))\}.$$

Then we have the following:

Proposition 4.3.1. *Let G be an arbitrary group and f be an orientation homomorphism. Then $\mathcal{U}_{g,f} \subseteq H \subseteq N_{\mathcal{U}}(\mathcal{U}_{g,f})$. In particular, if G is a periodic group, then $H = \mathcal{U}_{g,f}(\mathbb{Z}G)$.*

Proof. We need to prove only the second inclusion, i.e. $H \subseteq N_{\mathcal{U}}(\mathcal{U}_{g,f})$. Let $h \in H$ and $u \in \mathcal{U}_{g,f}$. Then $hh^f \in Z_2(\mathcal{U})$ and $uu^f = c \in C(\mathcal{U}) = Z_1(\mathcal{U})$, so $u^f = cu^{-1}$. Let $v = h^{-1}uh$. We conclude that $h^{-f}vv^fh^f = h^{-f}h^{-1}uhh^fu^f = (hh^f)^{-1}u(hh^f)cu^{-1} = [(hh^f)^{-1}, u]c \in C(\mathcal{U})$. It follows that $vv^f \in C(\mathcal{U})$. This means that $v \in \mathcal{U}_{g,f}$ and therefore, $h \in N_{\mathcal{U}}(\mathcal{U}_{g,f})$. When G is periodic, Theorem 2.2.4 says that $H = \mathcal{U}_{g,f}$. \square

Let

$$H_1 = \{u \in \mathcal{U}(\mathbb{Z}G) \mid uu^f \in \tilde{Z}(\mathcal{U}(\mathbb{Z}G))\}.$$

Recalling Theorem 4.2.6, we obtain another equivalent definition of the generalized unitary units of an integral group ring $\mathbb{Z}G$ when G is a periodic group as follows:

Corollary 4.3.2. *Let G be a periodic group. Then $H_1 = \mathcal{U}_{g,f}(\mathbb{Z}G)$.*

Since $\tilde{Z}(\mathcal{U}(\mathbb{Z}G)) \subseteq H_1$, we obtain the following:

Corollary 4.3.3. *Let G be a periodic group and f be any orientation homomorphism. Then $\tilde{Z}(\mathcal{U}(\mathbb{Z}G)) \subseteq \mathcal{U}_{g,f}(\mathbb{Z}G)$. In particular, $\tilde{Z}(\mathcal{U}(\mathbb{Z}G)) \subseteq N_{\mathcal{U}(\mathbb{Z}G)}(G)$.*

The question of when equality holds in Corollary 4.3.3 is settled by the following:

Theorem 4.3.4. *Let G be a periodic group and f be an orientation homomorphism. Then the following are equivalent:*

- (1) $\tilde{Z}(\mathcal{U}(\mathbb{Z}G)) = \mathcal{U}_{g,f}$;
- (2) $G = T$;
- (3) G is either a Hamiltonian 2-group or a torsion Abelian group;
- (4) $\tilde{Z}(\mathcal{U}(\mathbb{Z}G)) = \mathcal{U}(\mathbb{Z}G)$.

Proof. (1) \implies (2). Since $\mathcal{U}_{g,f}(\mathbb{Z}G) = \tilde{Z}(\mathcal{U}(\mathbb{Z}G))$, we have $G \subseteq \tilde{Z}(\mathcal{U}(\mathbb{Z}G))$. Therefore, $G = T$ by Corollary 4.2.3.

(2) \implies (3). If $G(= T)$ is nonabelian, then we are in case (1) of Theorem 4.2.1.

(3) \implies (4). If G is a Hamiltonian 2-group, then $\mathcal{U}(\mathbb{Z}G) = \pm G = \tilde{Z}(\mathcal{U}(\mathbb{Z}G))$. If G is abelian, then $\mathcal{U}(\mathbb{Z}G) = \mathcal{C}(\mathcal{U}(\mathbb{Z}G)) = \tilde{Z}(\mathcal{U}(\mathbb{Z}G))$.

(4) \implies (1). The result follows immediately from Corollary 4.3.3.

□

Note that in both cases of (3) of Theorem 4.3.4, $\mathcal{U}(\mathbb{Z}G) = \mathcal{U}_{g,f}(\mathbb{Z}G)$ and we obtain the following:

Corollary 4.3.5. *Let G be a periodic group. If $\mathcal{U}(\mathbb{Z}G) \neq \mathcal{U}_{g,f_1}(\mathbb{Z}G)$ for some orientation homomorphism f_1 , then there exists $u \in \mathcal{U}_{g,f_1}$ such that $u \notin \tilde{Z}(\mathcal{U}(\mathbb{Z}G))$, i.e. $\tilde{Z}(\mathcal{U}(\mathbb{Z}G)) \subset \mathcal{U}_{g,f_1}(\mathbb{Z}G)$. Furthermore, for all orientation homomorphisms f , $\tilde{Z}(\mathcal{U}(\mathbb{Z}G)) \subset \mathcal{U}_{g,f}(\mathbb{Z}G)$. In particular, $\tilde{Z}(\mathcal{U}(\mathbb{Z}G)) \subset N_{\mathcal{U}(\mathbb{Z}G)}(G)$.*

Remark 4.3.6. *It is possible that $\mathcal{U}(\mathbb{Z}G) \neq \mathcal{U}_{g,f_1}(\mathbb{Z}G)$ for some f_1 , but $\mathcal{U}(\mathbb{Z}G) = \mathcal{U}_{g,f_2}(\mathbb{Z}G)$ for another f_2 .*

For example, let us take $G = D_8 = \langle a, b \mid a^4 = b^2 = 1; bab = a^{-1} \rangle$.

(1) We first consider the trivial orientation homomorphism. Let u be a central unit. Then $u \in \mathcal{U}(\mathbb{Z} \langle a \rangle)$ by Bovdi and Sehgal [15], so u is a trivial unit by Higman's Theorem. Therefore, $N_{\mathcal{U}(\mathbb{Z}D_8)}(D_8) = \pm D_8 \neq \mathcal{U}(\mathbb{Z}D_8)$.

(2) Now we take $f : D_8 \rightarrow \pm 1$, such that $f(a) = 1$ and $f(b) = -1$. Theorem 2.2.18 (1) implies that $\mathcal{U}(\mathbb{Z}D_8) = \mathcal{U}_{g,f}(\mathbb{Z}D_8)$.

Chapter 5

The N-Centre of the Unit Group of an Integral Group Ring

5.1 Introduction

Let n be an integer. Two elements x, y in a group G n -commute if

$$(xy)^n = x^n y^n \text{ and } (yx)^n = y^n x^n,$$

see Baer [5]. A group is n -abelian if any two elements n -commute. In [6], Baer introduced the n -centre $Z(G, n)$ of a group G as the set of those elements which n -commute with every element in the group. Later Kappe and Newell [33] proved that $(ax)^n = a^n x^n$ for all $x \in G$ implies $(xa)^n = x^n a^n$ for all $x \in G$, and vice versa. Thus only one of the n -commutativity conditions suffices to define the n -centre $Z(G, n)$.

The n -centre, which can readily be seen to be a characteristic subgroup, shares many properties with the centre, some of which already have been explored in Baer [5]. For example, if the central quotient of a group is (locally) cyclic, then the group is abelian. Similarly, it follows by Corollary 1 in Baer [6] that a group is n -abelian if the quotient modulo its n -centre is (locally) cyclic. In [33], Kappe and Newell

shed further light on these similarities by investigating various characterizations and embedding properties of the n -centre. They characterized the n -centre as the margin of the n -commutator word $(xy)^n y^{-n} x^{-n}$, and their result yields some interesting connections with a conjecture of Hall on margins.

In this chapter, we investigate the n -centre of the unit group of an integral group ring $\mathbb{Z}G$ for a periodic group G . It is well known that the 2-centre of a group is equal to its centre. It turns out that the 3-centre of the unit group of an integral group ring of a periodic group also coincides with the centre of that unit group (Theorem 5.2.2). Our main result is to give a complete characterization of the n -centre of the unit group $\mathcal{U}(\mathbb{Z}G)$ for any integer n . We prove that this n -centre (for all $n \geq 2$) coincides with either the centre $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ of the unit group or the second centre $Z_2(\mathcal{U}(\mathbb{Z}G))$ of the unit group (Theorem 5.3.6). In view of Theorem 3.1 in [4], we obtain that the n -centre (for all $n \geq 2$) is either the centre $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ or the product of the centre and torsion hypercentral units, CT , when G is a finite group.

5.2 Basic Results and Notations

We first introduce some basic definitions and notations. Then we recall some fundamental results which will be needed later in this chapter. Other notations follow Kappe and Newell [33].

Let

$$S_1(G, n) = \{a \in G \mid (ax)^n = a^n x^n \forall x \in G\}$$

and

$$S_2(G, n) = \{a \in G \mid (xa)^n = x^n a^n \forall x \in G\}$$

Baer first defined the n -centre in [5] as

$$Z(G, n) = S_1(G, n) \cap S_2(G, n).$$

However, Kappe and Newell proved that $S_1(G, n) = S_2(G, n)$ ([33], Theorem 2.1).

Thus only one of the n -commutativity conditions suffices to define the n -centre.

The following proposition collects various facts about the elements in the n -centre. Note that $Z(G, 1) = Z(G, 0) = G$.

Proposition 5.2.1. ([33], Lemma 2.2) *Let $a \in Z(G, n)$. Then*

- (1) $[a^{n-1}, x^n] = 1$ for all $x \in G$;
- (2) $a \in Z(G, 1 - n)$ (Therefore always $Z(G, n) = Z(G, 1 - n)$);
- (3) $[a^n, x] = [a, x]^n = [a, x^n]$ for all $x \in G$;
- (4) $1 = [a, x^{n(1-n)}] = [a^{n(1-n)}, x] = [a, x]^{n(1-n)} = [a^n, x^{1-n}]$ for all $x \in G$;
- (5) $a^n \in Z(G, n - 1)$.

It can be easily seen by the definition that the 2-centre of a group coincides with its centre. Even a better result can be obtained when we investigate the 3-centre of the unit group $\mathcal{U}(\mathbb{Z}G)$ of an integral group ring of a periodic group G . We will show that the 3-centre $Z(\mathcal{U}(\mathbb{Z}G), 3)$ of the unit group also coincides with its centre $\mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ (sometimes denoted by $Z_1(\mathcal{U}(\mathbb{Z}G))$). In the next section, a characterization of $Z(\mathcal{U}(\mathbb{Z}G), n)$ will be obtained for all n .

Theorem 5.2.2. *Let G be a periodic group. Then*

$$Z(\mathcal{U}(\mathbb{Z}G), 3) = Z(\mathcal{U}(\mathbb{Z}G), 2) = \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$$

The following proposition due to Kappe and Newell is needed in the proof of Theorem 5.2.2.

Proposition 5.2.3. ([33], Theorem 4.3) *Let G be a group. Then*

$$Z(G, 3) = \{a \in R_2(G) \mid a^3 \in \mathcal{C}(G)\} \text{ and } Z(G, 3) \subseteq Z_3(G)$$

Here $Z_m(G)$ is the m -th centre of G and $R_m(G) = \{a \in G \mid [a, {}_m x] = 1 \forall x \in G\}$ denotes the set of right m -Engel elements, where

$$[x, {}_m y] = [[x, {}_{m-1} y], y] \text{ and } [x, {}_1 y] = [x, y].$$

Now we are ready to prove Theorem 5.2.2.

Proof. Recall that $Z(\mathcal{U}(\mathbb{Z}G), 3) \subseteq Z_3(\mathcal{U}(\mathbb{Z}G))$ by Proposition 5.2.3 and also that $Z_3(\mathcal{U}(\mathbb{Z}G)) = Z_2(\mathcal{U}(\mathbb{Z}G))$ and $Z_2^2(\mathcal{U}(\mathbb{Z}G)) \subseteq \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ by Theorem 4.2.6 and (4.2.2). It follows that for all $u \in Z(\mathcal{U}(\mathbb{Z}G), 3)$, $u^2 \in \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$. Also note that $u^3 \in \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ by Proposition 5.2.3. Thus $u \in \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ and $Z(\mathcal{U}(\mathbb{Z}G), 3) \subseteq \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$. We are done. \square

5.3 The Characterization of the N-Centre of the Unit Group of an Integral Group Ring

In this section, we investigate the n -centre of the unit group of an integral group ring for $n \geq 4$. We first characterize periodic Q^* -groups as precisely those periodic groups which contain a noncentral element lying in the 4-centre of $\mathcal{U}(\mathbb{Z}G)$. Then we turn our attention to studying the set of all torsion units in $Z(\mathcal{U}(\mathbb{Z}G), n)$. Our main result is Theorem 5.3.6, which gives a complete characterization of the n -centre of the unit group of an integral group ring for any periodic group.

A group G is said to be a Q^* -group if G has an Abelian normal subgroup A of index 2 which has an element a of order 4 such that for all $h \in A$ and all $g \in G \setminus A$, $g^2 = a^2$ and $g^{-1}hg = h^{-1}$. We note that finite Q^* -groups have played a significant role in work by Arora and Passi [4] (see also [3]), where they are characterized as precisely those groups G with the property that $\mathcal{U}_1(\mathbb{Z}G)$ is of central height 2. Such groups also appear in a paper by Williamson [59], who showed that Q^* groups are exactly those groups containing a noncentral element a which has finitely many conjugates in $\mathcal{U}(\mathbb{Z}G)$. Recently, Parmenter [46] showed that a weaker conjugation condition also characterizes these groups. For our purpose, we characterize these groups by means of the 4-centre of the unit group.

Theorem 5.3.1. *Let G be a periodic group. Then the following are equivalent:*

- (1) G is a Q^* -group;
- (2) G contains a noncentral element a such that $a \in Z(\mathcal{U}(\mathbb{Z}G), 4)$;
- (3) G contains a noncentral element a such that $a \in Z(\mathcal{U}(\mathbb{Z}G), n)$ for some $n \geq 4$.

To prove Theorem 5.3.1, we need the following results. The first one is proved by Parmenter in [46] (Theorem 1.2).

Proposition 5.3.2. *Let G be a periodic group. Then the following are equivalent:*

- (1) G contains a noncentral element a with the property that given any unit u in $\mathcal{U}(\mathbb{Z}G)$, there exists a positive integer $n = n(u)$ such that $u^n a u^{-n}$ belongs to G .
- (2) G is a Q^* -group.

The following proposition establishes a relationship between the 4-centre and the second centre of the unit group of an integral group ring.

Proposition 5.3.3. *Let G be a periodic group. Then $Z_2(\mathcal{U}(\mathbb{Z}G)) \subseteq Z(\mathcal{U}(\mathbb{Z}G), 4)$.*

Proof. Let $u \in Z_2(\mathcal{U}(\mathbb{Z}G))$ and $v \in \mathcal{U}(\mathbb{Z}G)$. Then we have $[u, v] \in \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ (*) and $u^2 \in Z_2^2(\mathcal{U}(\mathbb{Z}G)) \subseteq \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ (**) by the proof of Theorem 4.2.6 and (4.2.2). It follows that

$$[u, v]^2 = [u, v]u(vu^{-1}v^{-1}) = u[u, v](vu^{-1}v^{-1}) = u^2vu^{-2}v^{-1} = 1.$$

Therefore,

$$uvuv = uvu^{-1}v^{-1}vu^2v = [u, v]v^2u^2 \quad \text{since } u^2 \in \mathcal{C}(\mathcal{U}(\mathbb{Z}G)).$$

Consequently,

$$(uv)^4 = (uvuv)(uvuv) = ([u, v]v^2u^2)([u, v]v^2u^2) = [u, v]^2u^4v^4 = u^4v^4.$$

This leads to $u \in Z(\mathcal{U}(\mathbb{Z}G), 4)$ and we are done. \square

Now we are ready to prove Theorem 5.3.1.

Proof. (1) \implies (2) If G is a Q^* -group, then G has an Abelian subgroup A of index 2 which has an element a of order 4 such that for all $h \in A$ and all $g \in G \setminus A$, $g^2 = a^2$ and $g^{-1}hg = h^{-1}$. We claim that a is a noncentral element and belongs to $Z_2(\mathcal{U}(\mathbb{Z}G))$. Therefore, Proposition 5.3.3 implies that (2) is true.

It is obvious that a is noncentral. To see that $a \in Z_2(\mathcal{U}(\mathbb{Z}G))$, let us recall Theorem 2.2.18 (1) which guarantees that $\mathcal{U}(\mathbb{Z}G) = \mathcal{U}_{g,f}$ where $G = \langle A, b \rangle$ and $f : G \rightarrow \pm 1$ is the orientation homomorphism such that $\text{Ker}(f) = A$ and $f(b) = -1$.

It follows that for any $u = a_1 + a_2b$, $u^f = a_1^* - a_2a^2b$ and $u^{-1} = u^f c$ where c is a central unit. Now we have

$$[a, u] = (aua^{-1})u^{-1} = (a_1 + a_2a^2b)(a_1^* - a_2a^2b)c = (a_1a_1^* - a_2a_2^*a^2)c \in \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$$

Hence $a \in Z_2(\mathcal{U}(\mathbb{Z}G))$ and we are done.

(2) \implies (3). Immediate.

(3) \implies (1) Suppose $g \in Z(\mathcal{U}(\mathbb{Z}G), n) \setminus \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$. For $u \in \mathcal{U}(\mathbb{Z}G)$, Proposition 5.2.1(4) says that

$$[g, u^{n(1-n)}] = [g, u]^{n(1-n)} = [g^{n(1-n)}, u] = 1$$

Hence $u^{n(n-1)}gu^{-n(n-1)} = g \in G$ for all $u \in \mathcal{U}(\mathbb{Z}G)$ and Proposition 5.3.2 gives the desired result. \square

We can now obtain a different version of Proposition 5.3.2.

Corollary 5.3.4. *Let G be a periodic group. Then the following are equivalent:*

- (1) G is a Q^* -group;
- (2) G contains a noncentral element a such that for any unit $u \in \mathcal{U}(\mathbb{Z}G)$, $u^4au^{-4} = a$.

Proof. We need to verify only (1) \implies (2). By Theorem 5.3.1, G contains a noncentral element a such that $a \in Z(\mathcal{U}(\mathbb{Z}G), 4)$. It follows that for $u \in \mathcal{U}(\mathbb{Z}G)$, Proposition 5.2.1 (3) implies that

$$[a, u^4] = [a, u]^4 = [a^4, u] = 1$$

for $a^4 \in Z(\mathcal{U}(\mathbb{Z}G), 3) = \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ by Proposition 5.2.1(5) and Theorem 5.2.2.

Hence $u^4au^{-4} = a \in G$ for all $u \in \mathcal{U}(\mathbb{Z}G)$ □

Now we turn to characterizing the n -centre of the unit group. We first study the set of all torsion elements of the n -centre.

Theorem 5.3.5. *Let G be a periodic group and $T_n = T(Z(\mathcal{U}(\mathbb{Z}G), n)) = \{x \in Z(\mathcal{U}(\mathbb{Z}G), n) \mid x \text{ is of finite order and } \text{aug}(x) = 1\}$. Then for all $n \geq 2$,*

(1) T_n is a characteristic subgroup of $Z(\mathcal{U}(\mathbb{Z}G), n)$. Moreover,

$$T_n = Z(\mathcal{U}(\mathbb{Z}G), n) \cap G,$$

(2) If $u \in Z(\mathcal{U}(\mathbb{Z}G), n)$, then $[u, v] \in T_n$ for all $v \in \mathcal{U}(\mathbb{Z}G)$,

(3) $Z(\mathcal{U}(\mathbb{Z}G), n) \subseteq N_{\mathcal{U}(\mathbb{Z}G)}(G)$ and $Z^2(\mathcal{U}(\mathbb{Z}G), n) \subseteq T_n \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$,

(4) $T_n \subseteq T(Z_2(\mathcal{U}(\mathbb{Z}G)))$. Moreover, $T_4 = T(Z_2(\mathcal{U}(\mathbb{Z}G)))$,

(5) $Z(\mathcal{U}(\mathbb{Z}G), n) \subseteq Z_2(\mathcal{U}(\mathbb{Z}G))$. Moreover, $Z(\mathcal{U}(\mathbb{Z}G), 4) = Z_2(\mathcal{U}(\mathbb{Z}G))$.

Proof. (1) Referring to Theorem 5.2.2, we need to consider only the situation for $n \geq 4$ because central units of finite order are trivial (Sehgal [57], p46, Corollary 1.7). Note that if $a \in T_n$, then always $a^{-1} \in T_n$ since $o(a^{-1}) = o(a) < \infty$ and $a^{-1} \in Z(\mathcal{U}(\mathbb{Z}G), n)$. To prove T_n is a subgroup, we only need to show that if $a, b \in T_n$, then $ab \in T_n$, i.e. $o(ab) < \infty$. We will do it by using induction.

Let $n = 4$ and $a, b \in T_4$. Suppose that $o(a) = l, o(b) = m$. Thus

$$(ab)^{4lm} = (a^4b^4)^{lm} = a^{4lm}b^{4lm} = 1 \text{ (since } a^4, b^4 \in Z(\mathcal{U}(\mathbb{Z}G), 3) = \mathcal{C}(\mathcal{U}(\mathbb{Z}G))) \text{.}$$

Therefore, $ab \in T_4$. Consequently, T_4 is a subgroup.

Suppose that for $n = k > 3$, T_k is a subgroup of $Z(\mathcal{U}(\mathbb{Z}G), k)$.

Now consider that $n = k + 1$. For $a, b \in T_{k+1} \subseteq Z(\mathcal{U}(\mathbb{Z}G), k + 1)$, observe that $(ab)^{k+1} = a^{k+1}b^{k+1}$. Since $a^{k+1}, b^{k+1} \in Z(\mathcal{U}(\mathbb{Z}G), k)$ by Proposition 5.2.1 (5) and both have finite order, we conclude $a^{k+1}, b^{k+1} \in T_k$. It follows from the inductive assumption on T_k that $a^{k+1}b^{k+1} \in T_k$. As a consequence, $o((ab)^{k+1}) = o(a^{k+1}b^{k+1}) < \infty$, so $o(ab) < \infty$. This means that T_{k+1} forms a subgroup. We have proved that T_n is a subgroup of $Z(\mathcal{U}(\mathbb{Z}G), n)$ for every integer $n \geq 2$.

It can be easily seen that the subgroup T_n is a characteristic subgroup. Hence, since $Z(\mathcal{U}(\mathbb{Z}G), n)$ is a normal subgroup of the unit group $\mathcal{U}(\mathbb{Z}G)$ so is T_n . It follows from Bovdi [9] that $T_n \triangleleft G$. Therefore, $T_n = Z(\mathcal{U}(\mathbb{Z}G), n) \cap G$.

(2) Let $u \in Z(\mathcal{U}(\mathbb{Z}G), n)$ and $v \in \mathcal{U}(\mathbb{Z}G)$. Since $Z(\mathcal{U}(\mathbb{Z}G), n)$ is a normal subgroup of $\mathcal{U}(\mathbb{Z}G)$, we observe that $vu^{-1}v^{-1} \in Z(\mathcal{U}(\mathbb{Z}G), n)$; therefore, $[u, v] = uvu^{-1}v^{-1} \in Z(\mathcal{U}(\mathbb{Z}G), n)$. Moreover

$$[u, v]^{n(n-1)} = ([u, v]^{n(1-n)})^{-1} = 1 \text{ by Proposition 5.2.1 (4) .}$$

Hence, $[u, v] \in T_n$ as desired.

(3) The first statement follows directly from (1) and (2). Observing that

$$Z^2(\mathcal{U}(\mathbb{Z}G), n) \subseteq N_{\mathcal{U}(\mathbb{Z}G)}^2(G) \subseteq GC(\mathcal{U}(\mathbb{Z}G)) \text{ (Sehgal [58], Proposition 9.5),}$$

we easily obtain $Z^2(\mathcal{U}(\mathbb{Z}G), n) \subseteq T_n \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$.

(4) Suppose that for some $n \geq 2$ there exists $a \in T_n$ such that $a \notin T(Z_2(\mathcal{U}(\mathbb{Z}G)))$, thus a is a noncentral group element. According to Theorem 5.3.1, G is a Q^* -group. Next we show that this a is a special element of order 4 in G , as given in the definition of Q^* - groups. Observing the proof of Proposition 5.3.2, we find that if

$g \in G$, then either

(i) $\langle a, g \rangle$ is Abelian

or

(ii) $\langle a, g \rangle \cong Q_8$, the group of quaternions.

Setting $A = C_G(a) \subseteq G$ and $g \notin A$, we obtain that $\langle a, g \rangle \cong Q_8$, thus $a^2 = g^2$.

(Since a is not central, such a $g \notin A$ does exist). It follows that a has order 4.

For any $h \in A$, $g \notin A$, we have $hg \notin A$. Therefore, $\langle a, hg \rangle \cong Q_8$. It follows

that $g^2 = a^2 = hghg$, and so $ghg^{-1} = h^{-1}(*).$ We also note that if $k \notin A$, then

$gag^{-1} = a^{-1} = kak^{-1}$. It follows that $ag^{-1}k = g^{-1}ka$ and $g^{-1}k \in C_G(a) = A$,

and so A is of index 2 in G . Condition $(*)$ tells us that A is Abelian; therefore the

element a is a special element as we claimed. However we showed in the proof of

Theorem 5.3.1 that $a \in T(Z_2(\mathcal{U}(\mathbb{Z}G)))$. This contradiction leads to the first result.

Moreover, recalling Proposition 5.3.3 which gives $T(Z_2(\mathcal{U}(\mathbb{Z}G))) \subseteq T_4$, we obtain

that $T(Z_2(\mathcal{U}(\mathbb{Z}G))) = T_4$.

(5) Let $u \in Z(\mathcal{U}(\mathbb{Z}G), n)$ and $v \in \mathcal{U}(\mathbb{Z}G)$. Then $[u, v] \in T_n$ by (2); there-

fore, $[u, v] \in T(Z_2(\mathcal{U}(\mathbb{Z}G)))$ by (4). It follows that $u \in Z_3(\mathcal{U}(\mathbb{Z}G))$ and therefore,

$Z(\mathcal{U}(\mathbb{Z}G), n) \subseteq Z_3(\mathcal{U}(\mathbb{Z}G))$. Since $Z_3 = Z_2$ (Theorem 4.2.6), we conclude that

$Z(\mathcal{U}(\mathbb{Z}G), n) \subseteq Z_2(\mathcal{U}(\mathbb{Z}G))$. In particular, $Z(\mathcal{U}(\mathbb{Z}G), 4) \subseteq Z_2(\mathcal{U}(\mathbb{Z}G))$. Now Propo-

sition 5.3.3 finishes the proof. \square

Now we give a complete characterization of the n -centre of the unit group.

Theorem 5.3.6. *Let G be a periodic group. Then*

$$Z(\mathcal{U}(\mathbb{Z}G), n) = \begin{cases} \mathcal{U}(\mathbb{Z}G) & \text{for } n = 0 \text{ or } 1 \\ Z_2(\mathcal{U}(\mathbb{Z}G)) & \text{for } n = 4k \text{ or } 4k + 1, k \geq 1 \\ C(\mathcal{U}(\mathbb{Z}G)) & \text{for } n = 4k + 2 \text{ or } 4k + 3, k \geq 0 \end{cases}$$

Proof. The first equality is obvious.

Now we prove that $Z_2(\mathcal{U}(\mathbb{Z}G)) \subseteq Z(\mathcal{U}(\mathbb{Z}G), 4k)$ and $Z_2(\mathcal{U}(\mathbb{Z}G)) \subseteq Z(\mathcal{U}(\mathbb{Z}G), 4k+1)$ for all $k \geq 1$. Combined with Theorem 5.3.5(5), we have done the second part.

Let $u \in Z_2(\mathcal{U}(\mathbb{Z}G))$ and $v \in \mathcal{U}(\mathbb{Z}G)$. Then $u \in Z(\mathcal{U}(\mathbb{Z}G), 4)$ by Proposition 5.3.3, and therefore $u^4 \in \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$ by Proposition 5.2.1 (5) and Theorem 5.2.2. It follows that

$$(uv)^{4k} = ((uv)^4)^k = (u^4v^4)^k = u^{4k}v^{4k}.$$

This forces $u \in Z(\mathcal{U}(\mathbb{Z}G), 4k)$, thus $Z_2(\mathcal{U}(\mathbb{Z}G)) \subseteq Z(\mathcal{U}(\mathbb{Z}G), 4k)$.

Similarly,

$$(uv)^{4k+1} = (uv)(uv)^{4k} = uvu^{4k}v^{4k} = u^{4k+1}v^{4k+1}$$

This means that $Z_2(\mathcal{U}(\mathbb{Z}G)) \subseteq Z(\mathcal{U}(\mathbb{Z}G), 4k+1)$.

Next suppose that $n = 4k + 2$ or $4k + 3$, $k \geq 1$. First let us consider $n = 4k + 2$. Note that $Z(\mathcal{U}(\mathbb{Z}G), 4k + 2) \subseteq Z_2$ by Theorem 5.3.5 (5) and therefore, $Z(\mathcal{U}(\mathbb{Z}G), 4k + 2) \subseteq Z(\mathcal{U}(\mathbb{Z}G), 4k) \cap Z(\mathcal{U}(\mathbb{Z}G), 4k + 1)$ by the above. Recall that if an element is contained in 3 consecutive n -centres, then it must be a central element (see Kappe and Newell [33]). We are done. For completeness, we include a proof. Let $a \in Z(\mathcal{U}(\mathbb{Z}G), 4k + 2) \cap Z(\mathcal{U}(\mathbb{Z}G), 4k + 1) \cap Z(\mathcal{U}(\mathbb{Z}G), 4k)$ and $u \in \mathcal{U}(\mathbb{Z}G)$. Then

$$\begin{aligned}
(au)^{4k+2} &= a^{4k+2}u^{4k+2} \\
(au)^{4k+2} &= (au)^{4k+1}(au) = a^{4k+1}u^{4k+1}(au) \\
(au)^{4k+2} &= (au)^{4k}(au)^2 = a^{4k}u^{4k}(au)^2
\end{aligned}$$

Combining the first two equations, we obtain

$$au^{4k+1} = u^{4k+1}a$$

Combining the last two equations, we arrive at

$$au^{4k} = u^{4k}a$$

Now we conclude $au = ua$ and $a \in \mathcal{C}(\mathcal{U}(\mathbb{Z}G))$.

Similar arguments work for the case of $n = 4k + 3$. □

In view of Arora and Passi ([4], Theorem 3.1), we obtain the following corollary:

Corollary 5.3.7. *Let G be a finite group. Then*

$$Z(\mathcal{U}(\mathbb{Z}G), n) = \begin{cases} \mathcal{U}(\mathbb{Z}G) & \text{for } n = 0 \text{ or } 1 \\ T(Z_2(\mathcal{U}))\mathcal{C}(\mathcal{U}(\mathbb{Z}G)) & \text{for } n = 4k \text{ or } 4k + 1, k \geq 1 \\ \mathcal{C}(\mathcal{U}(\mathbb{Z}G)) & \text{for } n = 4k + 2 \text{ or } 4k + 3, k \geq 0 \end{cases}$$

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